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A Multiobjective Optimization Perspective on the Stability of Economic MPC

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Abstract: We interpret economic MPC as a scheme that trades off economic performance and stability. We use this notion to design an economic MPC controller that exploits the inherent robustness of a stable auxiliary MPC controller to enhance economic performance. Specifically, we incorporate a flexible stabilizing constraint to the economic MPC formulation that preserves stability of the auxiliary controller. We use multiobjective optimization concepts to argue that the dual variable of the stabilizing constraint can be interpreted as a *price of stability* and we establish an equivalence between the proposed controller and regularized economic MPC controllers. We demonstrate that nontrivial gains in economic performance can be achieved without compromising stability.

Keywords: economic, stability, MPC, multiobjective, price.

1. BASIC NOTATION AND SETTING

Consider a dynamic system of the form

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \Re^{n_x}$, $u_k \in \Re^{n_u}$, and $f : \Re^{n_x} \times \Re^{n_u} \to \Re^{n_x}$ is the system mapping. We assume that the system has an equilibrium point (x_{ss},u_{ss}) satisfying

$$x_{ss} = f(x_{ss}, u_{ss}). \tag{2}$$

We use the notation $\{x_k, u_k\}_t^{t+T}$ to describe a trajectory $x_k,\; k = t, ..., t + T$ and $u_k,\; k = t, ..., t + T - 1$. We say that a trajectory $\{x_k, u_k\}_t^{t+T}$ is feasible if it satisfies $x_{t+T} = x_{ss}$ and $x_k \in \mathcal{X}, u_k \in \mathcal{U}$. The sets $\mathcal{X} \subseteq \Re^{n_x}$ and $\mathcal{U} \subseteq \Re^{n_u}$ are assumed to be compact and contain the equilibrium point. If a trajectory $\{x_k, u_k\}_t^{t+T}$ is computed at time τ we denote this as $\{x_{k|\tau}, u_{k|\tau}\}_t^{t+T}$ and we define the compact notation $x_{t|t} = x_t, u_{t|t} = u_t$ and the vector $\mathbf{u}_T := (u_0, ..., u_{T-1})$.

We use the notation of Diehl et al. (2011) to define the admissible set in $T + 1$ steps as the joint set of initial states x_0 and control trajectories $\mathbf{u}_T$ giving rise to a set of feasible system trajectories. Formally,

$$\mathcal{W}_T = \{(x_0, \mathbf{u}_T) \mid x_{k+1} = f(x_k, u_k), x_k \in \mathcal{X},\; u_k \in \mathcal{U}, \text{ for } k = 0, ..., T-1, \text{ and } x_T = x_{ss}\}. \tag{3}$$

We define the set of admissible states as

$$\mathcal{Z}_T := \{\, x_0 \mid \exists\, \mathbf{u}_T \text{ s.t. } (x_0, \mathbf{u}_T) \in \mathcal{W}_T \}. \tag{4}$$

Consider now the following *tracking and economic value functions*, respectively, evaluated along a feasible trajectory $\{x_{k|t}, u_{k|t}\}_t^{t+T}$ computed at time t,

$$V_t^{tr} := \sum_{k=t}^{t+T-1} L^{tr}(x_{k|t} - x_{ss}, u_{k|t} - u_{ss}) \tag{5a}$$

$$V_t^{ec} := \sum_{k=t}^{t+T-1} L^{ec}(x_{k|t}, u_{k|t}). \tag{5b}$$

We assume that the stage cost $L^{tr} : \Re^{n_x} \times \Re^{n_u} \to \Re$ is a mapping satisfying $L^{tr}(x - x_{ss}, u - u_{ss}) = 0$ if and only if $x = x_{ss}$ and $u = u_{ss}$ hold and it is positive otherwise. In other words, the tracking stage cost is a positive definite function. For simplicity, we drop the dependence on x_{ss} and u_{ss} from the notation and use the compact form $L^{tr}(x, u)$. Positive definiteness implies that the tracking stage cost is bounded below and we assume further that it is bounded above. The economic stage cost is given by $L^{ec} : \Re^{n_x} \times \Re^{n_u} \to \Re$, and we assume this to be bounded below and above.

Consider now that at time instant $t + 1$, we have a trajectory $\{\bar{x}_{k|t+1}, \bar{u}_{k|t+1}\}_{t+1}^{t+1+T}$ with value function

$$\bar{V}_{t+1}^{tr} := \sum_{k=t+1}^{t+T} L^{tr}(\bar{x}_{k|t+1}, \bar{u}_{k|t+1}). \tag{6}$$

We assume that $\{\bar{x}_{k|t+1}, \bar{u}_{k|t+1}\}_{t+1}^{t+1+T}$ is the solution of the tracking MPC problem (MPC-T),

$$\min_{z_k, v_k} \sum_{k=0}^{T-1} L^{tr}(z_k, v_k) \tag{7a}$$

$$\text{s.t. } z_{k+1} = f(z_k, v_k),\; k = 0, ..., T-1 \tag{7b}$$

$$z_k \in \mathcal{X}, v_k \in \mathcal{U},\; k = 0, ..., T-1 \tag{7c}$$

$$z_T = x_{ss} \tag{7d}$$

$$z_0 = x_{t+1}. \tag{7e}$$

Here, z_k, v_k are internal optimization variables. The trajectory $\{\bar{x}_{k|t+1}, \bar{u}_{k|t+1}\}_{t+1}^{t+1+T}$ is optimal for MPC-T and thus feasible. As discussed by Mayne et al. (2000) it follows

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that:

$$\bar{V}_{t+1}^{tr} - V_t^{tr} \leq -L^{tr}(x_t, u_t). \tag{8}$$

Condition (8) is sufficient for stability because $L^{tr}(\cdot,\cdot)$ is a positive definite function and thus the tracking value function qualifies as a Lyapunov function.

Stability cannot be guaranteed when the stage cost $L^{tr}(\cdot,\cdot)$ is replaced by an arbitrary economic stage cost $L^{ec}(\cdot,\cdot)$. The reason is that the economic stage cost might not be a positive definite function and thus the economic value function does not qualify as a Lyapunov function. Strategies to avoid this issue include the addition of regularization terms guaranteeing that the regularized economic value function becomes a Lyapunov function, as is done by Huang et al. (2011). Alternatively, one might need to rely on system-specific properties (e.g., strong duality, dissipativity, or turnpike properties) to guarantee stability. These approaches are discussed by Diehl et al. (2011), Grüne (2013), Grüne and Stieler (2014), and Angeli et al. (2012). In this work, we construct an economic MPC controller (MPC-E) that exploits the inherent robustness properties of MPC-T to enhance economic performance.

2. MPC-E CONTROLLER

Because $\{\bar{x}_{k|t+1}, \bar{u}_{k|t+1}\}_{t+1}^{t+1+T}$ is optimal for MPC-T it gives the *best* progress in terms of the tracking value function and the structure of MPC-T automatically guarantees stability. A key observation that we make in this work is that stability can still be guaranteed for any feasible but suboptimal trajectory (with respect to MPC-T) that we denote as $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ and that satisfies,

$$V_{t+1}^{tr} \leq \bar{V}_{t+1}^{tr} + \sigma(V_t^{tr} - \bar{V}_{t+1}^{tr}), \tag{9}$$

for any scalar $\sigma \in [0, 1)$. Here,

$$V_{t+1}^{tr} := \sum_{k=t+1}^{t+T} L^{tr}(x_{k|t+1}, u_{k|t+1}). \tag{10}$$

To see that stability is implied by (9), we first note that this condition is equivalent to

$$V_{t+1}^{tr} - V_t^{tr} \leq (1-\sigma)(\bar{V}_{t+1}^{tr} - V_t^{tr}). \tag{11}$$

This follows by adding $-V_t^{tr}$ on both sides of (9). By using the lower bound (8) we have that (9) implies that

$$V_{t+1}^{tr} - V_t^{tr} \leq -(1-\sigma)L^{tr}(x_t, u_t). \tag{12}$$

This condition is sufficient for stability because the function $(1-\sigma)L^{tr}(\cdot,\cdot)$ is positive definite for $\sigma \in [0, 1)$. We thus refer to condition (9) as the *stabilizing constraint*. We emphasize that (9) implies (12) but not the other way around.

A suboptimal trajectory $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ (with respect to MPC-T) satisfying (9) can be obtained by solving the economic MPC problem (MPC-E),

$$\min_{z_k, v_k} \sum_{k=0}^{T-1} L^{ec}(z_k, v_k) \tag{13a}$$

$$\text{s.t. } z_{k+1} = f(z_k, v_k),\ k = 0, ..., T-1 \tag{13b}$$

$$z_k \in \mathcal{X}, v_k \in \mathcal{U},\ k = 0, ..., T-1 \tag{13c}$$

$$z_0 = x_{t+1} \tag{13d}$$

$$z_T = x_{ss} \tag{13e}$$

$$\sum_{k=0}^{T-1} L^{tr}(z_k, v_k) \leq \epsilon_{t+1}(\sigma), \tag{13f}$$

where

$$\epsilon_{t+1}(\sigma) := \bar{V}_{t+1}^{tr} + \sigma(V_t^{tr} - \bar{V}_{t+1}^{tr}) \tag{14}$$

is a parameter (function of parameter σ). From the solution of MPC-E we set

$$V_{t+1}^{tr} = \sum_{k=0}^{T-1} L^{tr}(z_k, v_k) = \sum_{k=t+1}^{t+T} L^{tr}(x_{k|t+1}, u_{k|t+1}). \tag{15}$$

Consequently, (13f) is the *stabilizing constraint* (9). We now formally define the MPC-E controller.

MPC-E Controller

(0) Given $x_0 \in \mathcal{X}$ and $\sigma \in [0, 1)$, set $t \leftarrow 0$ and $\epsilon_0(\sigma) \leftarrow +\infty$.
(1) Solve MPC-E for state x_t and $\epsilon_t(\sigma)$, evaluate V_t^{tr}, and set $u_t \leftarrow v_0$.
(2) Implement u_t, and let system evolve to $x_{t+1} = f(x_t, u_t)$.
(3) Solve MPC-T for state x_{t+1}, and evaluate $\bar{V}_{t+1}^{tr}$.
(4) Set $\epsilon_{t+1}(\sigma) \leftarrow \bar{V}_{t+1}^{tr} + \sigma(V_t^{tr} - \bar{V}_{t+1}^{tr})$.
(5) Set $t \leftarrow t+1$ and RETURN to Step 1.

We define the MPC-E control law resulting from the above scheme as $h(x_t, \sigma)$.

Inherent Robustness: The stabilizing constraint (9) is designed to *exploit the inherent robustness of MPC-T to optimize economic performance*. Discussions on inherent robustness are presented by Pannocchia et al. (2011) and Santos and Biegler (1999). If a suboptimal trajectory (with respect to MPC-T) with tracking value V_{t+1}^{tr} is used instead of the optimal trajectory with tracking value $\bar{V}_{t+1}^{tr}$, we have that

$$\begin{aligned} V_{t+1}^{tr} - V_t^{tr} &= \bar{V}_{t+1}^{tr} - V_t^{tr} + (V_{t+1}^{tr} - \bar{V}_{t+1}^{tr}) \\ &\leq -L^{tr}(x_t, u_t) + (V_{t+1}^{tr} - \bar{V}_{t+1}^{tr}). \end{aligned} \tag{16}$$

Here, we used the lower bound (8) and $(V_{t+1}^{tr} - \bar{V}_{t+1}^{tr})$ is the suboptimality error with respect to MPC-T. Consequently, we can improve economic performance and maintain stability as long as the suboptimality error satisfies the condition

$$(V_{t+1}^{tr} - \bar{V}_{t+1}^{tr}) \leq \alpha L^{tr}(x_t, u_t) \tag{17}$$

with $\alpha < 1$. Note also that the larger $L^{tr}(x_t, u_t)$ is, the more flexibility (inherent robustness) we have to improve economic performance. One would be tempted to replace the stabilizing constraint (9) with the suboptimality error condition (17) or directly impose the condition,

$$V_{t+1}^{tr} - V_t^{tr} \leq -(1+\beta)L^{tr}(x_t, u_t), \tag{18}$$

for some $\beta > 0$, as is proposed by Maree and Imsland (2014) (see Assumption 3 in their work). Establishing lower bounds for α and upper bounds for β under which (17) and (18) are guaranteed to hold, respectively, is complicated and might require trial and error. In particular, Maree and Imsland (2014) do not propose a procedure to determine an upper bound for β. The reason behind this complication is that both conditions (17) and (18) are derived by using the lower bound (8). The sufficient stability condition (9), on the other hand, *can hold only for* $\sigma \in [0, 1)$. This will be shown in the next section. These observations also imply that there is more flexibility than the one provided by the stage cost $L^{tr}(x_t, u_t)$.

Feasibility and Convexity: In the following, we make the blanket assumption that MPC-T is feasible for any $x_t \in \mathcal{Z}_T$. Moreover, we assume that any solution of MPC-T and MPC-E satisfies a Slater condition. We first focus on the special case in which the system mapping $f(\cdot, \cdot)$ is linear, the stage functions $L^{tr}(\cdot, \cdot)$ and $L^{ec}(\cdot, \cdot)$ are convex, and the sets $\mathcal{X}$ and $\mathcal{U}$ are convex. Convexity and a Slater condition guarantee that MPC-T and MPC-E have unique primal and dual solutions (strong duality holds). These properties will simplify the derivation and explanation of several properties of MPC-E. In Section 4 we present safeguards that enable us to generalize these results to nonconvex settings.

3. STABILITY AND PROPERTIES OF MPC-E

In the following discussion, we use the compact representation of MPC-E (13),

$$\min V^{ec}_{t+1} \tag{19a}$$
$$\text{s.t. } (13b) - (13d) \tag{19b}$$
$$V^{tr}_{t+1} \leq \epsilon_{t+1}(\sigma). \quad (\lambda_{t+1}) \tag{19c}$$

Here, λ_{t+1} is the Lagrange multiplier (dual variable) of the stabilizing constraint (19c). For a fixed σ, we denote the tracking and economic value functions resulting from the solution of (19) as $V^{tr}_{t+1}(\sigma)$ and $V^{ec}_{t+1}(\sigma)$, respectively. We define the control action $u_{t+1}(\sigma)$ and multiplier $\lambda_{t+1}(\sigma)$ as functions of the parameter σ.

The MPC-E problem has the following properties.

Lemma 1. For fixed x_{t+1}, MPC-E is feasible for all $\sigma \in [0, 1)$, and $V^{ec}_{t+1}(\sigma)$ is a nonincreasing function of $\sigma \in [0, 1)$.

Proof: From feasibility of MPC-T we have that $\bar{V}^{tr}_{t+1}$ exists and is finite. MPC-T and MPC-E have the same constraint set (except for (13f)) but MPC-E must be feasible with respect to (13f) for $\sigma = 0$. Moreover, from the structure of (13f), the lower bound (8), and the positive definiteness of $L^{tr}(\cdot, \cdot)$ we have that $\sigma(V^{tr}_t - \bar{V}^{tr}_{t+1}) \geq \sigma L^{tr}(x_t, u_t) \geq 0$ holds for all $\sigma \in (0, 1)$. Consequently, the term $\sigma(V^{tr}_{t+1} - \bar{V}^{tr}_{t+1})$ is a positive relaxation of the stabilizing constraint (13f), and thus the feasible region of MPC-E for $\sigma = 0$ is contained in the feasible region defined for $\sigma \in (0, 1)$. We thus have feasibility of MPC-E for all $\sigma \in [0, 1)$.

We can establish the second result as follows. Pick some parameter $\sigma_1 \in (0, 1)$. By construction we have that $\epsilon_{t+1}(\sigma_1) > 0$. Define the functions $f_0 := V^{ec}_{t+1}(\sigma_1)$ and $f_1 := V^{tr}_{t+1}(\sigma_1) - \epsilon_{t+1}(\sigma_1)$ and construct the optimization problem

$$\min f_0 \ \text{ s.t. } \ f_1 + \kappa \leq 0 \quad (\lambda),$$

which is an analog of (19) with fixed parameter $\kappa = 0$. Denote the function values at the solution as $f_0(\kappa)$ and $f_1(\kappa)$ and the multiplier as $\lambda(\kappa)$. By construction we have that $\lambda(\kappa) \geq 0$. Under convexity of MPC-E we have from Theorem 6.1. by Rockafellar (1993) that

$$f_0(\kappa) \geq f_0(0) + \lambda(0)\kappa,$$

holds for any κ. Now pick κ satisfying $0 < \kappa < \epsilon_{t+1}(\sigma_1)$. This is equivalent to picking some $\sigma_2 < \sigma_1$ because $\epsilon_{t+1}(\sigma_2) < \epsilon_{t+1}(\sigma_1)$ holds for $\sigma_2 < \sigma_1$. Because $\lambda(0) \geq 0$ and $\kappa > 0$, we have that $f_0(\kappa) \geq f_0(0)$. This implies that $V^{ec}_{t+1}(\sigma_2) \geq V^{ec}_{t+1}(\sigma_1)$ for $\sigma_2 < \sigma_1$. The result follows. □

Stability Price: By construction of MPC-E, we have that $\lambda_{t+1}(\sigma) \geq 0$; and from duality we have that

$$\lambda_{t+1}(\sigma) = \begin{cases} -\dfrac{\partial V^{ec}_{t+1}(\sigma)}{\partial \epsilon_{t+1}(\sigma)} & \text{IF } V^{tr}_{t+1} = \epsilon_{t+1}(\sigma) \\ 0 & \text{IF } V^{tr}_{t+1} < \epsilon_{t+1}(\sigma). \end{cases} \tag{20}$$

Consequently, for fixed σ (and thus fixed $\epsilon_{t+1}(\sigma)$), we have that the value functions are *conflicting* if and only if $\lambda_{t+1}(\sigma) > 0$. In other words, as we increase σ we have that $V^{ec}_{t+1}(\sigma)$ decreases, as stated in Lemma 1. We can thus see that MPC-E can be interpreted as a multiobjective (MO) optimization problem that seeks a trade off economic performance and rate of decay of the tracking function (and thus convergence to the equilibrium point). In our setting, the trade-off is determined by the parameter σ, which relaxes the stabilizing constraint. Moreover, the dual variable $\lambda(\sigma)$ of the stabilizing constraint can be interpreted as a *stability price*. In other words, the stability price is a measure of the resistance imposed by the stabilizing constraint on economic performance. If the objectives are not conflicting (possible in some applications) we have that $\lambda_{t+1}(\sigma) = 0$ and thus the stabilizing constraint becomes irrelevant. In such a case one can minimize the economic value function freely without worrying about stability.

We can formalize the MO interpretation by defining the MPC-MO problem

$$\min \{V^{ec}_{t+1}, V^{tr}_{t+1}\} \tag{21a}$$
$$\text{s.t. } (13b) - (13d).$$

The set of solutions of the MPC-MO is comprised of weak Pareto solutions and Pareto solutions which we now define.

Definition 1. (**Weak Pareto Optimality**) A trajectory $\{x_{k|t+1}, u_{k|t+1}\}^{t+1+T}_{t+1}$ with corresponding value functions $V^{tr}_{t+1}, V^{ec}_{t+1}$ is *weak Pareto optimal for MPC-MO* if there does not exist an alternative trajectory with value functions $\hat{V}^{tr}_{t+1}, \hat{V}^{ec}_{t+1}$ satisfying $\hat{V}^{tr}_{t+1} < V^{tr}_{t+1}$ and $\hat{V}^{ec}_{t+1} < V^{ec}_{t+1}$.

Definition 2. (**Pareto Optimality**) A trajectory $\{x_{k|t+1}, u_{k|t+1}\}^{t+1+T}_{t+1}$ with value functions $V^{tr}_{t+1}, V^{ec}_{t+1}$ is *Pareto optimal for MPC-MO* if there does not exist an alternative trajectory with value functions $\hat{V}^{tr}_{t+1}, \hat{V}^{ec}_{t+1}$ satisfying $\hat{V}^{tr}_{t+1} \leq V^{tr}_{t+1}$ and $\hat{V}^{ec}_{t+1} < V^{ec}_{t+1}$ or $\hat{V}^{ec}_{t+1} \leq V^{ec}_{t+1}$ and $\hat{V}^{tr}_{t+1} < V^{tr}_{t+1}$.

In other words, weak Pareto solutions can be strictly improved in one objective while the other one remains constant (e.g., points along a vertical or horizontal line) while Pareto optimal solutions cannot be improved strictly in both objectives (e.g., points along a diagonal). Spanning the range $\sigma \in [0, 1)$ defines the so-called *Pareto front* and this procedure corresponds to the so-called ϵ-constrained method reviewed by Miettinen (1999).

Proposition 1. The trajectory $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ obtained from MPC-E is a weak Pareto optimal for MPC-MO for any $\sigma \in [0, 1)$.

Proof: Assume that the trajectory $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ with corresponding value functions $V_{t+1}^{tr}, V_{t+1}^{ec}$ solves MPC-E but it is not weakly Pareto optimal. Then, there exists an alternative trajectory with tracking and economic value functions $\hat{V}_{t+1}^{tr}, \hat{V}_{t+1}^{ec}$ satisfying $\hat{V}_{t+1}^{tr} < V_{t+1}^{tr}$ and $\hat{V}_{t+1}^{ec} < V_{t+1}^{ec}$. This implies that $\hat{V}_{t+1}^{tr} < V_{t+1}^{tr} \leq \epsilon_{t+1}(\sigma)$ and thus the alternative trajectory is feasible for MPC-E. However, because $\hat{V}_{t+1}^{ec} < V_{t+1}^{ec}$ and V_{t+1}^{ec} is optimal for MPC-E we have a contradiction. □

A solution trajectory $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ obtained from MPC-E can only be guaranteed to be weak Pareto optimal. One can verify, however, if this solution is also Pareto optimal. Consider the alternative problems

$$\min V_{t+1}^{ec} \tag{22a}$$
$$\text{s.t. } (13\text{b}) - (13\text{d})$$
$$V_{t+1}^{tr} \leq \hat{V}_{t+1}^{tr} \tag{22b}$$

and,

$$\min V_{t+1}^{tr} \tag{23a}$$
$$\text{s.t. } (13\text{b}) - (13\text{d})$$
$$V_{t+1}^{ec} \leq \hat{V}_{t+1}^{ec}. \tag{23b}$$

Proposition 2. A trajectory $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ is Pareto optimal for MPC-MO if and only if it is the solution of the alternative problems (22) and (23) with $\hat{V}_{t+1}^{tr}, \hat{V}_{t+1}^{ec}$ evaluated at $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$.

Proof: See Theorem 3.2.2 by Miettinen (1999). □

Connection with Regularized Economic MPC: When strong duality holds (in our setting this is true under convexity), one can relate the solution of the MPC-E problem (19) to that of the weighted problem

$$\min\ (1 - \omega_{t+1})V_{t+1}^{ec} + \omega_{t+1}V_{t+1}^{tr} \tag{24}$$
$$\text{s.t. } (13\text{b}) - (13\text{d}),$$

with weight parameter $\omega_{t+1} \in [0, 1]$.

Proposition 3. The trajectory $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ obtained from MPC-E (19) with associated multiplier $\lambda_{t+1}(\sigma)$ is a solution trajectory of the weighted problem (24) with $\omega_{t+1} := \frac{\lambda_{t+1}(\sigma)}{1+\lambda_{t+1}(\sigma)}$ and $\omega_{t+1} \in [0, 1]$.

Proof: If strong duality holds, the solution of MPC-E is the solution of

$$\min V_{t+1}^{ec} + \lambda_{t+1}(\sigma)V_{t+1}^{tr} \tag{25}$$
$$\text{s.t. } (13\text{b}) - (13\text{d})$$

for fixed $\lambda_{t+1}(\sigma)$. Rescaling the objective function of (25) by $\frac{1}{1+\lambda_{t+1}(\sigma)}$ does not change the solution trajectory. The result follows from the definition of ω_{t+1}. □

We thus have that the incorporation of the stabilizing constraint (13f) can be interpreted as the addition of a tracking regularization term, in the spirit of the work of Huang et al. (2011), Subramanian et al. (2014), and Maree and Imsland (2014). Weighted regularization is in general cumbersome because the weight needs to be adjusted at each time t (if at all updated). Maree and Imsland (2014) do this using an automatic weighting procedure that selects the maximum weight that preserves stability at each time t. Using the stabilizing constraint proposed in this work eliminates the need for such a procedure.

Note that the weight obtained from MPC-E is an implicit function of σ and thus we can express it as $\omega_{t+1}(\sigma)$. We also note also that if the objectives are not conflicting we have that $\lambda_{t+1}(\sigma) = 0$ implies $\omega_{t+1}(\sigma) = 0$.

Theorem 1. The equilibrium point x_{ss} under the control law $h(x_t, \sigma)$ is an asymptotically stable equilibrium with region of attraction $\mathcal{Z}_T$ for any $\sigma \in [0, 1)$.

Proof: From feasibility of MPC-T and using Lemma 1 we have feasibility of MPC-E. The stabilizing constraint (13f) evaluated at $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ guarantees that $V_{t+1}^{tr} - V_t^{tr} \leq -(1-\sigma)L^{tr}(x_t, u_t)$ for all t. Because $L^{tr}(x_t, u_t)$ is a positive definite function and is bounded from above, we have that V_t^{tr} is a Lyapunov function. □

Corollary 1. $\lim_{t\to\infty} \epsilon_t(\sigma) = 0$ and $\lim_{t\to\infty} V_t^{ec}(\sigma) = T \cdot L^{ec}(x_{ss}, u_{ss})$ for all $\sigma \in [0, 1)$.

Proof: From stability we have that $\lim_{t\to\infty} V_t^{tr} = 0$, and from (8) we have that $\lim_{t\to\infty} \bar{V}_t^{tr} = 0$ and $\lim_{t\to\infty} \epsilon_t(\sigma) = 0$. From (13f) and the fact that $L^{tr}(\cdot, \cdot)$ is a positive definite function we have that the only feasible solution of MPC-E in the limit is $z_k = x_{ss}$, $v_k = u_{ss}$. □

Instability: From (12) we can see that $\sigma = 1$ implies that $V_{t+1}^{tr}(1) \leq V_t^{tr}$ and, if tracking and economic objectives are conflicting (e.g., $\lambda_{t+1}(1) > 0$), we have that $V_{t+1}^{tr}(1) = V_t^{tr}$. Consequently, the tracking value function is not strictly decreasing, and stability cannot be guaranteed.

Connection with other MO controllers: The MPC-E controller differs from recent multiobjective MPC controllers proposed in the literature. In the multiobjective controllers presented by Vito De and Scattolini (2007) and Bemporad and Muñoz de la Peña (2009) all the value functions are required to be Lyapunov functions. In our setting we only the tracking value function to be a Lyapunov function. In the utopia-tracking controller presented by Zavala and Flores-Tlacuahuac (2012), the authors define multiple value functions $\Phi^j(\cdot)$ with $j = 1, ..., N_\Phi$ with ideal values $\bar{\Phi}^j$ obtained from the solution of

$$\min_{z_k, v_k}\ \Phi^j(\{z_k, v_k\}_0^T) \tag{26a}$$
$$\text{s.t. } z_{k+1} = f(z_k, v_k),\ k = 0, ..., T-1 \tag{26b}$$
$$z_k \in \mathcal{X}, v_k \in \mathcal{U},\ k = 0, ..., T-1 \tag{26c}$$
$$z_0 = x_{t+1} \tag{26d}$$
$$z_T = x_{ss}, \tag{26e}$$

for $j = 1, ..., N_\Phi$. The ideal values $\bar{\Phi}^j$ define the so-called utopia point. The authors then design an MPC controller that minimizes the utopia-tracking function,

$$\left(\sum_{j=1}^{N_\Phi} \left|\Phi^j(\{z_k, v_k\}_0^T) - \bar{\Phi}^j\right|^p\right)^{1/p}, \tag{27}$$

subject to the constraints (26b)-(26e). The utopia-tracking function (27) is augmented with a Lagrangian penalization term to derive a Lyapunov function, as proposed by Diehl et al. (2011). The utopia-tracking controller permits the handling of many objective functions but relies on strong duality to guarantee stability. The MPC-E controller proposed in this work can also be used to handle many objectives. This can be done replacing the objective function in (13) with (27) and by defining the economic value function

$$V_{t+1}^{ec} := \left(\sum_{j=1}^{N_\Phi} |\Phi^j(\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}) - \bar{\Phi}^j|^p\right)^{1/p}, \tag{28}$$

where the ideal values $\bar{\Phi}^j$ are computed at each time step. Stability of this MPC-E controller is enforced using the stabilizing constraint (13f). In Section 4 we show that strong duality is not needed for this MPC-E controller, which provides an advantage over the utopia-tracking controller of Zavala and Flores-Tlacuahuac (2012).

Connection with Lyapunov-Based Controllers: MPC-E is related to Lyapunov-based MPC controllers, as the one proposed by Heidarinejad et al. (2012), but differs in how feasibility in ensured. In Lyapunov-based controllers the descent of a Lyapunov function is imposed directly in the controller formulation and feasibility with respect to this constraint and state constraints is enforced by designing the level set of the Lyapunov function. The stabilizing constraint (9) of MPC-E also implies descent of the Lyapunov function but feasibility is enforced by inheriting feasibility of MPC-T. The stabilizing constraint (9) of MPC-E can be used to add flexibility to Lyapunov-based controllers. In particular, the descent condition in the Lyapunov-based controller of Heidarinejad et al. (2012) corresponds to choosing $\sigma = 0$ for the stability constraint (11) and thus can limit economic performance. We note, however, that the controller proposed by Heidarinejad et al. (2012) does not require descent of the Lyapunov function at each time instant, while MPC-E does.

4. CONSIDERATIONS FOR NONCONVEX CASE

Assume now that the system mapping $f(\cdot,\cdot)$, the stage functions $L^{tr}(\cdot,\cdot), L^{ec}(\cdot,\cdot)$, and the sets $\mathcal{X}$ and $\mathcal{U}$ are allowed to be nonconvex. Nonconvexity gives rise to several issues that need to be resolved. First note that it is possible that no global solution might be available for the MPC-T and MPC-E. This issue does not prevent us from finding a feasible (albeit local) solution for MPC-T. This thus implies that MPC-E is at least feasible for $\sigma = 0$ and this guarantees stability. One cannot guarantee, however, that MPC-E is feasible for all $\sigma \in [0, 1)$. Second, we cannot guarantee that $V_{t+1}^{ec}(\sigma)$ is a decreasing function of $\sigma \in [0, 1)$. This issue is related to the presence of discontinuities and nonconvex segments in the Pareto front. It is also well-known that, in the presence of nonconvexity, the weighted formulation (24) might not identify Pareto optimal points for certain values of the weight ω (see pp. 79-80 in the work of Miettinen (1999)).

We provide strategies to deal with nonconvexities in our control setting. First note that continuity of the Pareto front (with respect to parameter σ and thus $\epsilon_{t+1}(\sigma)$) can be guaranteed in a nonempty neighborhood of $\sigma = 0$ if MPC-E satisfies the so-called sufficient second-order conditions (SSOC). SSOC implies that the the solution is uniquely defined and that the solution is Lipschitz continuous in σ in a neighborhood of $\sigma = 0$. For a review of these concepts see Zavala (2008); Zavala and Anitescu (2010). The decreasing property of $V_{t+1}^{ec}(\sigma)$ with σ can be guaranteed locally if a solution exists at a fixed value of σ because $\lambda_{t+1}(\sigma) \geq 0$ by construction. Moreover, we can find a suitable $\sigma \in [0, 1)$ where a solution of MPC-E exists by solving the following optimization problem:

$$\min_{z_k, u_k, \sigma} \sum_{k=0}^{T-1} L^{ec}(z_k, v_k) \tag{29a}$$

$$\text{s.t. } z_{k+1} = f(z_k, v_k),\ k = 0, ..., T-1 \tag{29b}$$

$$z_k \in \mathcal{X}, v_k \in \mathcal{U},\ k = 0, ..., T-1 \tag{29c}$$

$$z_0 = x_{t+1} \tag{29d}$$

$$z_T = x_{ss} \tag{29e}$$

$$\sum_{k=0}^{T-1} L^{tr}(z_k, v_k) \leq \bar{V}_{t+1}^{tr} + \sigma(V_t^{tr} - \bar{V}_{t+1}^{tr}) \tag{29f}$$

$$0 \leq \sigma \leq 1 - \delta. \tag{29g}$$

Here $\delta \in (0, 1)$ is a small parameter. Problem (29) finds the trajectory $\{x_{k|t+1}, u_{k|t+1}\}_{t+1}^{t+1+T}$ and a feasible $\sigma \in [0, 1)$ that minimizes the economic objective. This problem is feasible because it admits $\sigma = 0$ as a solution. We thus have that, if σ is obtained by solving (29) at each time t, then Theorem 1 follows. We note that, in problem (29), σ acts as a slack variable that is increased to minimize the economic objective. Consequently, if $\sigma = 0$ is the solution of (29), we have an indication that MPC-E is infeasible for any $\sigma \in (0, 1)$.

Connection with Economic Lyapunov Framework: Under convexity and a Slater condition we have that strong duality holds and stability of MPC-E can also be established using the Lyapunov framework for economic MPC of Diehl et al. (2011). Consequently, the stabilizing constraint (13f) of MPC-E is not needed. In the nonconvex case, however, strong duality is no longer guaranteed to hold. In this more general case we can guarantee stability of MPC-E by solving (29) instead of (13) . This provides an advantage over the Lyapunov framework of Diehl et al. (2011).

5. CASE STUDY

We consider the nonlinear chemical reactor system,

$$c_{t+1}^A = c_t^A + h\left(\frac{Q_t}{V}(c_f^A - c_t^A) - k_r {c_t^A}^2\right) \tag{30a}$$

$$c_{t+1}^B = c_t^B + h\left(\frac{Q_t}{V}(c_f^B - c_t^B) + k_r {c_t^A}^2\right), \tag{30b}$$

where c_t^A, c_t^B are the states and Q_t is the control. The tracking and economic stage costs are given by

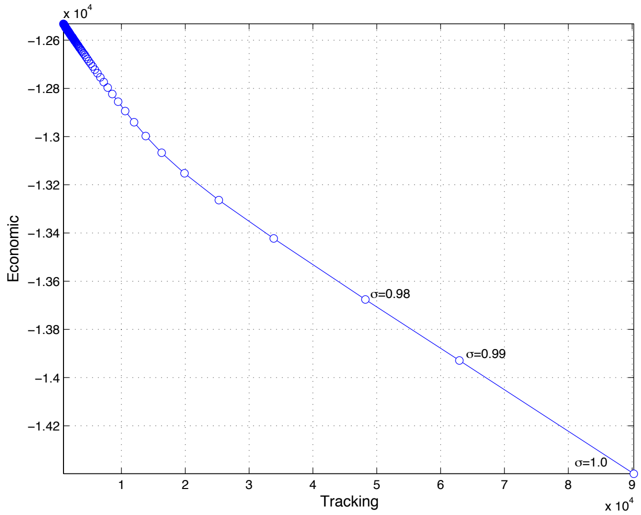

Fig. 1. Accumulated economic and tracking value functions for different values of σ.

$$L^{tr}(c_t^A, c_t^B, Q_t) = \frac{1}{2}|c_t^A - 1/2|^2 + \frac{1}{2}|c_t^B - 1/2|^2 + \frac{1}{2}|Q_t - 12|^2 \quad (31a)$$

$$L^{ec}(c_t^A, c_t^B, Q_t) = -2Q_t c_t^B + \frac{1}{2}Q_t. \quad (31b)$$

The system has an equilibrium point at $c_{ss}^A = c_{ss}^B = 1/2$ and $Q_{ss} = 12$. We consider a closed-loop study with a time horizon $T = 20$, $N = 100$ time steps, initial conditions $c_0^A = 0.4$ and $c_0^B = 0.2$, and we span the range $\sigma \in [0, 1]$. The system data is provided by Diehl et al. (2011) and an AMPL implementation of this study is available at `http://www.mcs.anl.gov/~vzavala/multiobjec.tgz`.

In Figure 1 we present the trade-off curve of accumulated economic and tracking value functions $Ec(\sigma) := \sum_{t=0}^{N} V_t^{ec}(\sigma)$ and $\sum_{t=0}^{N} V_t^{tr}(\sigma)$, respectively. We construct this curve by spanning the σ interval $[0, 1]$ in increments of 0.01. As can be seen, the value functions are conflicting and the economic value function quickly decays with σ as it approaches the stability boundary of $\sigma = 1$ (see points at $\sigma = 0.98, 0.99$). In other words, a slight relaxation of the stabilizing constraint can reach high gains in economic performance. Moreover, these gains can be achieved without compromising stability. The relative gap in economic performance $100 \cdot (Ec(0) - Ec(1))/Ec(0)$ is of 14%.

The smoothness of the trade-off curve of Figure 1 indicates that MPC-E is defined for different values of σ. We highlight, however, that this curve is not technically a Pareto front because the Pareto front of MPC-MO is defined at each time for the current state x_t. Constructing the Pareto fronts at each point in time is computationally intensive; consequently, we do not present them here.

In Figure 2 we present the tracking value functions $V_t^{tr}(\sigma)$. The tracking function is strictly decreasing for all $\sigma \in [0, 1)$ and eventually decays to zero and thus stability is achieved. For $\sigma = 1$ stability is not achieved, as predicted by Theorem 1. The speed of decay increases with decreasing σ. Stability can also be visualized in Figure 3 where we present the time profile for the reactant concentration c_t^A. The profile for $\sigma = 1$ does not converge to the equilibrium

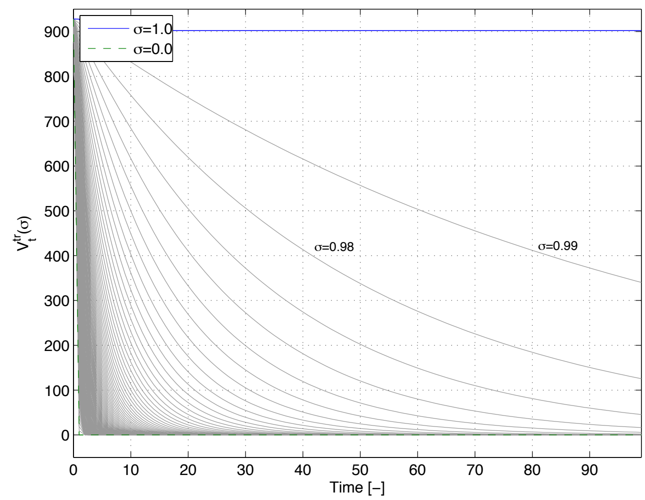

Fig. 2. Tracking function $V_t^{tr}(\sigma)$ for different values of σ.

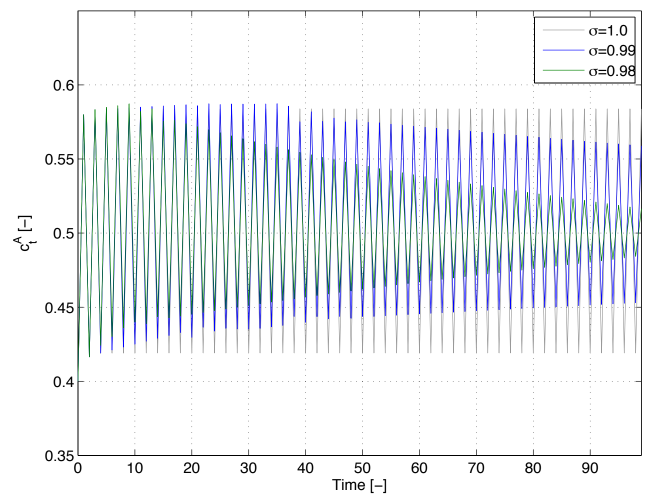

Fig. 3. Concentration profile c_t^A for $\sigma = 1, 0.99, 0.98$.

point (i.e., it is periodic) while for $\sigma = 0.99$ and $\sigma = 0.98$ the profiles exhibit periodicity at the beginning of the horizon and eventually decay to the equilibrium point x_{ss} (this is most notable for $\sigma = 0.98$). We highlight that we only show 100 time steps to illustrate the behavior of the dynamic profiles. The instability for $\sigma = 1$ reinforces the observation of Diehl et al. (2011) who note that strong duality does not hold for the reactor system and thus stability cannot be guaranteed. To overcome this limitation, Diehl et al. (2011) add the quadratic regularization term (31a) to the economic objective (31b). This is equivalent to solve the weighted MPC-E problem (24) with fixed weight. The stabilizing constraint of MPC-E provides a similar regularization effect but the weight is automatically adjusted at each time (see Proposition 3).

In Figure 4 we present time profiles for the stability prices $\lambda_t(\sigma)$. The prices increase as we decrease σ and this indicates that decreasing σ restricts economic performance. For $\sigma = 1$ we have that the price is zero, indicating that the stabilizing constraint does not restrict economic performance (i.e., no further gains can be achieved by relaxing the stabilizing constraint further). For each σ, the prices increase in time and eventually settle down because

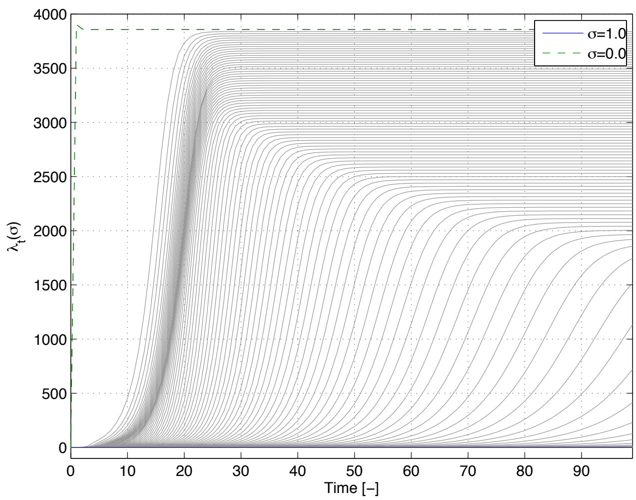

Fig. 4. Stability price $\lambda_t(\sigma)$ for different values of σ.

the system settles at x_{ss}. The settling value of the price is positive, implying that the system always has an economic incentive to leave the equilibrium point x_{ss} to enhance economic performance. We currently do not understand, however, why the settling level is different for different values of σ. This is counterintuitive because the system settles at the same equilibrium point in all cases.

6. CONCLUSIONS

We have presented an interpretation of economic MPC as that of a controller that trades off economic performance and stability. Using this notion, we derive an economic MPC controller that makes use of a flexible stabilizing constraint, and we use multiobjective optimization concepts to introduce the notion of *price of stability*. We demonstrate that nontrivial gains in economic performance can be obtained without compromising stability. As part of future work it is necessary to understand the dynamic behavior of the stability price and design controllers that balance economics, stability, and robustness. It is also possible to extend the analysis to include terminal regions instead of terminal constraints. Real-time implementations following the ideas in Biegler et al. (2015) and Diehl et al. (2002) are also desirable because the economic MPC controller requires the solution of two optimization problems.

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