Robustly Stable Economic NMPC for Non-Dissipative Stage Costs ⋆

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Abstract

We analyze the inherent robustness properties of an economic NMPC formulation in which the controller trades off rate of convergence and economic performance. We show that this controller is input-to-state practically stable under reasonable assumptions. Our formulation does not require dissipativity with respect to the stage costs being optimized, as is required by existing economic MPC formulations. Instead, our formulation enforces dissipation in the form of a Lyapunov inequality that is constructed by using traditional tracking cost terms. Consequently, the proposed approach can be applied to a wider range of systems. We also demonstrate that the controller provides high flexibility to optimize economic performance and remains robust in the face of disturbances.

Key words: Predictive control, economics, nonlinear control, robust stability.

1 Introduction

Model predictive control (MPC) has seen a variety of applications as it can naturally handle inequality constraints, multivariable interactions, and different cost functions. A survey of industrial applications of MPC is given in [26], and a thorough theoretical treatment of MPC is given in [29]. Nonlinear model predictive control (we use the term MPC to refer to the nonlinear case as well) has the added advantage of being able to capture nonlinear effects and thus provides higher accuracy across a wide range of states [11]. Fast MPC implementations in highly complex systems are enabled by noticing that an exact real-time solution of the associated nonlinear programming (NLP) problems is not required [25,41,37,40].

Input-to-state stability (ISS) is a powerful theoretical tool for analyzing and characterizing the robustness of control systems. The property was originally described for continuous time systems in [32] and was extended to discrete time systems in [18]. ISS has also been adapted to characterize the robustness of general MPC formulations [21,8]. Extensions of ISS such as input-to-state practical stability (ISpS) allow for wider scope of this tool.

Standard stability and robustness analysis tools need to be modified when general (economic) cost functions are used in MPC formulations. Asymptotic stability for economic MPC (eMPC) can be established for systems that satisfy strict dissipativity [3,28,23]. In such formulations, strict dissipativity requires the existence of a storage function which can be naturally obtained in certain types of physical systems (e.g., mechanical or energy systems) or that can be obtained if strong duality of the equilibrium point holds. Stability of economic MPC can also be guaranteed if the system satisfies the so-called turnpike property [4,5], even in the absence of terminal constraints [12,9]. Furthermore, it has been shown that dissipativity and turnpike properties are closely related [10]. We highlight that dissipativity and turnpike properties currently used in economic MPC are system-specific properties. Consequently, they cannot be guaranteed in general applications and can be difficult to check in practice. This is the case, for instance, in large-scale chemical processes such as polymer plants, separation systems, and pulp and paper plants [42,16,34,13]. To enable asymptotic stability in a more general setting, it is possible to regularize the economic cost function to make it strongly convex (e.g., by adding a tracking cost

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term) [17,33]. Regularization terms, however, are difficult to tune and can be conservative, limiting economic gains [2].

In [39] it is proposed to replace the regularization term with a stabilizing inequality constraint. The stabilizing constraint is derived by exploiting the inherent robustness margin of an auxiliary and asymptotically stable MPC controller. It is demonstrated that this approach (that which we call `eMPC-sc`) provides high flexibility to optimize economic performance while retaining stability. Moreover, it is shown that this approach is a special type of regularization-based and Lyapunov-based MPC approaches. In particular, `eMPC-sc` provides adaptive regularization through the constraint (as in trust-region schemes used in optimization). The `eMPC-sc` controller also differs from the Lyapunov-based approach (see [14]) in that feasibility of the stabilizing constraint can be guaranteed directly, while in the Lyapunov approach the feasible set for the states needs to be adjusted to ensure feasibility. A formulation for the tracking MPC case with uncertainty with a similar stabilizing constraint is provided in [1]. A closely related formulation is shown in [22], with the main difference being that we avoid explicitly deriving a weighting function. In this work, we analyze the robustness properties of `eMPC-sc`, and show that the resulting closed-loop system is (ISpS). We present small and large-scale studies to demonstrate the applicability of the approach.

The paper is structured as follows. In Section 2 we provide basic definitions and notation. In Section 3 we discuss the ISpS property and introduce a modified Lyapunov function that satisfies ISpS. The `eMPC-sc` formulation and theoretical properties are presented in Section 4. Case studies are presented in Section 6. The paper closes with general remarks and directions of future work.

2 Notation and Definitions

We consider the system:

$$x_{k+1} = f(x_k, u(x_k), w_k) \tag{1}$$

where $x \in \mathcal{X}$ is a vector of states, the set $\mathcal{X} \subset \mathbb{R}^{n_x}$ is closed and bounded, the control law $u(x) : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$ is a mapping of the current state, which in this work will be representative of a given MPC formulation, and $w \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ is a vector of disturbances. We use $|\cdot|$ as the 2-norm, $\mathbb{R}$ as the set of real numbers, $\mathbb{Z}$ as the set of integers, and the subscript $+$ to indicate the nonnegative counterparts. We define the truncated disturbance sequence at time $k \in \mathbb{Z}_+$ as $\mathbf{w}_k = [w_0, \dots, w_{k-1}, 0, \dots]$, and the full disturbance sequence $\mathbf{w} = [w_0, w_1, w_2, \dots]$. We also define the set $\mathcal{W} := \{\mathbf{w} : w_k \in \mathbb{W} \text{ for all } k = 0, 1, 2, \dots\}$. We note that the state at time k, x_k, is an implicit function of x_0 and $\mathbf{w}_k$ for a given control scheme, but these arguments are not shown for simplicity. We make the following basic assumptions and definitions.

Assumption 1 *(A) The set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is robustly positive invariant for $f(\cdot,\cdot,\cdot)$. That is, $f(x, u(x), w) \in \mathcal{X}$ holds for all $x \in \mathcal{X}$, $w \in \mathbb{W}$. (B) The set $\mathbb{W}$ is bounded and $||\mathbf{w}|| := \sup_{k \in \mathbb{Z}_+} |w_k|$. (C) f is uniformly continuous with respect to w.*

Assumption 1 is assumed to hold for the remainder of this work.

Definition 2 *(Comparison Functions). A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$ function and $\lim_{s \to \infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is a $\mathcal{K}$ function, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \to \infty} \beta(s, t) = 0$.*

Definition 3 *(Attractivity). The system (1) is attractive on $\mathcal{X}$ if $\lim_{k \to \infty} x_k = 0$ for all $x_0 \in \mathcal{X}$.*

Definition 4 *(Stable Equilibrium Point). The point $x = 0$ is called a stable equilibrium point of (1) if, for all $k_0 \in \mathbb{Z}_+$ and $\epsilon_1 > 0$, there exists $\epsilon_2 > 0$ such that $|x_{k_0}| < \epsilon_2 \Rightarrow |x_k| < \epsilon_1$ for all $k \geq k_0$.*

Definition 5 *(Asymptotic Stability). The system (1) is asymptotically stable on $\mathcal{X}$ if $\lim_{k \to \infty} x_k = 0$ for all $x_0 \in \mathcal{X}$ and $x = 0$ is a stable equilibrium point.*

We highlight that asymptotic stability is only possible for disturbances w_k that converge to a constant. See Appendix B of [29] for the preceding definitions.

Definition 6 *(ISS). The system (1) is input-to-state stable (ISS) on $\mathcal{X}$ if $|x_k| \leq \beta(|x_0|, k) + \gamma(||\mathbf{w}||)$ holds for all $x_0 \in \mathcal{X}$ and $k \geq 0$, where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$.*

We note that, in the nominal case (with no disturbances), ISS reduces to asymptotic stability. The following is a useful variant of ISS.

Definition 7 *(ISpS): Under Assumption 1, the system (1) is input-to-state practically stable (ISpS) on $\mathcal{X}$ if $|x_k| \leq \beta(|x_0|, k) + \gamma(||\mathbf{w}||) + c$ holds for all $x_0 \in \mathcal{X}$ and $k \geq 0$, where $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and $c \in \mathbb{R}_+$.*

The reader is referred to [21] for more details on these definitions. We highlight that ISpS is more flexible than ISS as the sequence is relaxed by a non-vanishing constant c. As a result, however, there is no guarantee of asymptotic stability in the nominal case. In Section 5.1 we will show that the constant c is bounded by a $\mathcal{K}_\infty$ function of $\hat{u} := \max_{u \in \mathbb{U}} |u|$ if the controller is designed appropriately.

3 ISpS with a Modified Lyapunov Function

We use the ISpS Lyapunov theorem of [21] with a modified Lyapunov function. Traditionally, the Lyapunov function is a function of the current state, $V : \mathbb{R}^n \to \mathbb{R}$. In this work, we analyze our proposed `eMPC-sc` controller using a modified Lyapunov function that is a function of the path, $V : \mathbb{Z}_{\geq 0, \leq K} \times \mathbb{R}^{n_w \times K} \times \mathbb{R}^{n_x} \to \mathbb{R}$ where the first argument is the time step, the second argument is the series of disturbances, and the third argument is the initial state. We first show that the modified Lyapunov function satisfies the ISpS properties established in [21]. This is formalized in the following result.

Theorem 8 *Let Assumption 1 hold. If the system* (1) *admits a function* $V(k, \mathbf{w}_k, x_0)$ *satisfying:*

$$\alpha_1(|x_k|) \leq V(k, \mathbf{w}_k, x_0) \leq \alpha_2(|x_k|) + c_1 \tag{2a}$$

$$V(k+1, \mathbf{w}_{k+1}, x_0) - V(k, \mathbf{w}_k, x_0)$$
$$\leq -\alpha_3(|x_k|) + \sigma(|w_k|) + c_2 \tag{2b}$$
$$\forall \ x_0 \in \mathcal{X}, \ \mathbf{w} \in \mathcal{W}, \ k \in \mathbb{Z}_+$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, *and* $c_1, c_2 \in \mathbb{R}_+$ *then the system is ISpS.*

Proof: See Appendix. □

As can be seen, the constants c_1, c_2 used in the Lyapunov function definition (2) relax the upper bound and the descent conditions of the traditional Lyapunov function used to enforce the more stringent ISS conditions. Consequently, we highlight that ISS is recovered for the special case in which $c_1 = c_2 = 0$.

4 MPC Formulations

Having established basic definitions and properties, we now introduce our economic MPC formulation `eMPC-sc` and related formulations to highlight advantages and disadvantages.

4.1 Tracking and Regularized eMPC Formulations

We first consider the nominal tracking MPC controller that solves the following NLP:

$$\min_{z_i, v_i} \quad \sum_{i \in \mathcal{N}} L^{tr}(z_i, v_i) \tag{3a}$$
$$\text{s.t.} \quad z_{i+1} = f(z_i, v_i, 0) \ \ \forall \, i \in \mathcal{N} \tag{3b}$$
$$z_0 = x_k \tag{3c}$$
$$z_i \in \mathbb{X}, \ v_i \in \mathbb{U} \ \ \forall \, i \in \mathcal{N} \tag{3d}$$
$$z_N = 0 \tag{3e}$$

where $z \in \mathbb{R}^{n_x}$ and $v \in \mathbb{R}^{n_u}$ are the predicted states and controls, respectively, the sets $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ and $\mathbb{U} \subset \mathbb{R}^{n_u}$ are compact and contain the origin in their interiors and $\mathcal{N} := \{0, ..., N-1\}$. Note that the set $\mathbb{X}$ simply describes the state inequalities in (3) and is distinct from, and not necessarily related to, the RPI set $\mathcal{X}$. This distinction is motivated by the observation that, in practice, it is desirable for a controller to still operate if the state is taken outside of $\mathbb{X}$ by a disturbance. The mapping $L^{tr} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}_+$ is the tracking stage cost penalizing deviations from the steady-state. At each time k, the NLP is solved for x_k, and the first control $u(x_k) := v_0$ is implemented to the system. The following assumption imposes a basic requirement on the nature of the tracking stage cost and other basic assumptions for tracking MPC formulations.

Assumption 9 *(A) There exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that* $\alpha_2(|x|) \geq L^{tr}(x, u) \geq \alpha_1(|x|)$ *for all* $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{U}^{n_u}$*. (B) A solution to (3) exists for all* $x_k \in \mathbb{X}$*. (C) The functions* $L^{tr}(\cdot, \cdot)$*, and* $f(\cdot, \cdot, \cdot)$ *are twice continuously differentiable.*

Definition 10 *Weak controllability is satisfied for a given MPC formulation if the computed control trajectory* v_i *satisfies*

$$\sum_{i \in \mathcal{N}} |v_i| \leq \alpha(|x|) \tag{4}$$

for some $\alpha \in \mathcal{K}_\infty$*.*

The upper bound $\alpha_2(|x|) \geq L^{tr}(x, u)$ holds if weak controllability [3] holds, because $|v_i| \leq \alpha(|x|)$ holds $\forall \, i$. In this work we state the definition of weak controllability as applying explicitly to the control trajectory found by the controller and not just some control trajectory that exists, since the objective term $v^T R v$ is not necessarily present in all MPC formulations.

The tracking stage cost usually has the form

$$L^{tr}(z, v) = z^T Q z + v^T R v \tag{5}$$

where Q, R are positive semidefinite matrices but other norms can also be used to satisfy Assumption 9A [43,27]. The following result is standard.

Theorem 11 *Under Assumption 9 and* $w_k = 0$ *for all* $k \in \mathbb{Z}_+$*, the closed-loop system under the tracking MPC controller is asymptotically stable for all* $x_0 \in \mathbb{X}$*.*

See Theorem 6.18 in [11] for proof. Note that, for $x_k \in \mathcal{X} \setminus \mathbb{X}$, problem (3) is infeasible and the controller fails. This is a major disadvantage of using hard state constraints that is rectified by using a penalty reformulation with soft constraints, as we show in the next section.

We now proceed to describe economic MPC formulations. We define (x_{ss}, u_{ss}) as the solution of the steady-state problem:

$$(x_{ss}, u_{ss}) := \underset{x,u}{\operatorname{argmin}} \; L^{ec}(x,u) \tag{6a}$$
$$\text{s.t. } x = f(x,u,0) \tag{6b}$$
$$x \in \mathbb{X},\; u \in \mathbb{U} \tag{6c}$$

where the mapping $L^{ec} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ is the economic stage cost. We assume, without loss of generality, that $(x_{ss}, u_{ss}) = (0, 0)$. The NLP solved by a standard economic MPC (eMPC) controller is:

$$\min_{z_i,v_i} \sum_{i\in\mathcal{N}} L^{ec}(z_i, v_i) \tag{7a}$$
$$\text{s.t. } z_{i+1} = f(z_i, v_i, 0) \;\; \forall\, i \in \mathcal{N} \tag{7b}$$
$$z_0 = x_k \tag{7c}$$
$$z_i \in \mathbb{X},\; v_i \in \mathbb{U} \;\; \forall\, i \in \mathcal{N} \tag{7d}$$
$$z_N = 0 \tag{7e}$$

Of course we expect better economic performance from this formulation compared to tracking MPC but asymptotic stability of eMPC is not guaranteed in general. This is because Assumption 9 is not fulfilled for general $L^{ec}(\cdot,\cdot)$. Stability of eMPC can be enforced by appending a tracking term that regularizes the economic objective. Under this approach, the regularized eMPC problem is:

$$\min_{z_i,v_i} \sum_{i\in\mathcal{N}} (L^{ec}(z_i, v_i) + \omega L^{tr}(z_i, v_i)) \tag{8a}$$
$$\text{s.t. } z_{i+1} = f(z_i, v_i, 0) \;\; \forall\, i \in \mathcal{N} \tag{8b}$$
$$z_0 = x_k \tag{8c}$$
$$z_i \in \mathbb{X},\; v_i \in \mathbb{U} \;\; \forall\, i \in \mathcal{N} \tag{8d}$$
$$z_N = 0 \tag{8e}$$

where $\omega \in \mathbb{R}_+$. We define the rotated stage cost $L^{rot}(x,u) := L^{ec}(x,u) + \omega L^{tr}(x,u) + \lambda(x) - \lambda(f(x,u,0)))$, where $\lambda : \mathbb{R}^{n_x} \to \mathbb{R}$ is a storage function. One choice for λ is to use the Lagrange multipliers of (6), so that $\lambda(x) - \lambda(f(x,u,0)) = \lambda(x - f(x,u,0))$ is locally nonnegative. We denote this approach as `eMPC-reg`.

Definition 12 *The system* (1) *is strictly dissipative with respect to supply rate* $L^{ec}(z_i, v_i) + \omega L^{tr}(z_i, v_i)$ *if there exists some* $\alpha \in \mathcal{K}_\infty$ *such that* $L^{rot}(x,u) \ge \alpha(|x|)\; \forall\, x \in \mathbb{R}^{n_x}$

As shown in [3,28], if strict dissipativity holds, then eMPC-reg is asymptotically stable. In the case of [3], it is shown that strict dissipativity holds if the steady-state problem (6) satisfies strong duality. When $L^{tr}(\cdot)$ is quadratic, weighting matrices Q and R in (5) can be found by applying the Gershgorin circle theorem to the Hessian of the rotated stage cost $\nabla^2 L^r(x,u)$, as is done in [17,2]. Unfortunately, finding and tuning weighting matrices is cumbersome for large problems. In particular, for a general NLP, $\nabla^2 L^r(x,u)$ must be checked at every $x \in \mathcal{X}$ and $u \in \mathbb{U}$ of (6) in order to ensure that the Hessian is always positive definite. Furthermore, the end result is often conservative in the sense that the regularization term may dominate the economic stage cost, thus limiting economic performance.

We note that asymptotic stability of tracking MPC or eMPC may be extended to ISS in the presence of nonzero disturbances by using a *relaxed reformulation* of the NLP [38]. The relaxed reformulation (see (11)) is essential to ensure a uniformly continuous value function, which provides a sufficient condition for robust stability (i.e., the cost function remains bounded under perturbations). The relaxed reformulation is achieved by softening state constraints with slack variables that are penalized in the objective function. We discuss this issue in more detail in the following subsection.

4.2 `eMPC-sc` *Formulation*

It has been recently suggested to replace the tracking regularization terms in the objective with a stabilizing constraint [39]. This controller solves the NLP:

$$\min_{z_i,v_i} \sum_{i\in\mathcal{N}} L^{ec}(z_i, v_i) \tag{9a}$$
$$\text{s.t. } z_{i+1} = f(z_i, v_i, 0) \;\; \forall\, i \in \mathcal{N} \tag{9b}$$
$$z_0 = x_k \tag{9c}$$
$$z_i \in \mathbb{X},\; v_i \in \mathbb{U} \;\; \forall\, i \in \mathcal{N} \tag{9d}$$
$$z_N = 0 \tag{9e}$$
$$\sum_{i\in\mathcal{N}} L^{tr}(z_i, v_i) - V(k-1, \mathbf{w}_{k-1}, x_0) \le -\delta\, L^{tr}(x_{k-1}, u_{k-1}) \tag{9f}$$

where $\delta \in (0, 1]$ is a scalar parameter. After the NLP is solved, we inject the control law $u(x_k) = v_0$ into the system and set

$$V(k, \mathbf{w}_k, x_0) := \sum_{i\in\mathcal{N}} L^{tr}(z_i^k, v_i^k), \tag{10}$$

where z_i^k, v_i^k is the solution of (9) at time k. We thus note that $V(k-1, \mathbf{w}_{k-1}, x_0)$ is the value function at time $k-1$.

Once the control is injected into the system we wait for it to evolve to x_{k+1} and use the value function $V(k, \mathbf{w}_k, x_0)$ in (9f) to solve (9) at x_{k+1}, and repeat the procedure. Note that, for the problem solved at time $k = 0$, the constraint (9f) may simply be excluded. Also, the formulation of the stabilizing constraint (9f) is slightly different from that used in [39]. The advantages of this formulation are that we do not require a solution to the tracking problem, and this formulation provides a looser constraint with the same stability properties.

We will prove that under this recursion the constraint (9f) forces the descent of $V(k, \mathbf{w}_k, x_0)$ explicitly. We also note that the value function $V(k, \mathbf{w}_k, x_0)$ is a function of the path since the solution of (9) depends on the previous value function. In Section 5 we will prove that, despite this, the value function is a Lyapunov function satisfying the properties of Theorem 8 and thus the system under `eMPC-sc` is ISpS. We also note that $\delta = 1$ in (9f) corresponds to the most constrained Lyapunov function, and δ approaching zero corresponds to the least constrained. The parameter δ is thus a tuning parameter that shapes closed-loop behavior. A large δ forces a faster approach to the steady state, and small δ allows for more economic flexibility.

We highlight that $V(k-1, \mathbf{w}_{k-1}, x_0)$ and $L^{tr}(x_{k-1}, u_{k-1})$ are fixed quantities in the NLP and thus the stabilizing constraint (9f) can be written as $\sum_{i\in\mathcal{N}} L^{tr}(z_i, v_i) \leq \Delta_k$ with $\Delta_k := V(k-1, \mathbf{w}_{k-1}, x_0) - \delta\, L^{tr}(x_{k-1}, u_{k-1})$. If we consider positive definite matrices Q and R, we have that $L^{tr}(z, v) = z^T Q z + v^T R v$ and the stabilizing constraint satisfies $\sum_{i\in\mathcal{N}}(z_i^T Q z_i + v_i^T R v_i) \leq \Delta_k$ and thus defines a trust-region constraint with radius Δ_k around the origin $(0, 0)$. This trust region defines a space that the controller can explore at time instant k to improve economic performance while preserving stability. Moreover, by dualizing the stabilizing constraint we can obtain formulation (8). Consequently, we can see that the stabilizing constraint acts as a regularization term. The weight ω, however, is determined by the optimal Lagrange multiplier of (9f) and thus changes at each time instant k (i.e., the weight is adaptive). The Lagrange multiplier can be interpreted as the *price of stability.* We highlight that this approach does not require dissipativity with respect to stage costs appearing in the objective (i.e. Definition 12 with $\omega = 0$) or turnpike properties, which contrasts with existing economic MPC formulations [3,4]. Because of this, the approach has wider applicability. For instance, because this approach does not require strong duality, any equilibrium point (x_{ss}, u_{ss}) can be used. For more details, see [39].

To establish ISpS for the proposed economic MPC controller we must ensure uniform continuity of the value function. This property is not guaranteed for the formulation (9). This can be achieved by softening the state constraints, as is done in [38]. To do this, we assume that $\mathbb{X}$ and $\mathbb{U}$ are expressed as constraints of the form $\mathbb{X} = \{x : g_x(x) \leq 0\}$ and $\mathbb{U} = \{u : g_u(u) \leq 0\}$. The `eMPC-sc` problem is thus:

$$\min_{z_i, v_i} \quad \sum_{i\in\mathcal{N}} L^{ec}(z_i, v_i) + \rho\left(\sum_{i\in\mathcal{N}} \xi_i^x + \xi^S + \xi^{ss,L} + \xi^{ss,U}\right) \tag{11a}$$

$$\text{s.t. } z_{i+1} = f(z_i, v_i, 0) \ \ \forall\, i \in \mathcal{N} \tag{11b}$$

$$z_0 = x_k \tag{11c}$$

$$g_x(z_i) \leq \xi_i^x \ \ \forall\, i \in \mathcal{N} \tag{11d}$$

$$g_u(v_i) \leq 0 \ \ \forall\, i \in \mathcal{N} \tag{11e}$$

$$-\xi^{ss,L} \leq z_N \leq \xi^{ss,U} \tag{11f}$$

$$\sum_{i\in\mathcal{N}} L^{tr}(z_i, v_i) - V(k-1, \mathbf{w}_{k-1}, x_0) \leq -\delta\, L^{tr}(x_{k-1}, u_{k-1}) + \xi^S \tag{11g}$$

$$\xi_i^x, \xi^S, \xi^{ss,L}, \xi^{ss,U} \geq 0, \tag{11h}$$

where $\xi_i^x, \xi^S, \xi^{ss,L}, \xi^{ss,U}$ are slack variables and $\rho \in \mathbb{R}_+$ is a penalty parameter. Finding a value for ρ for the formulations used in this paper is discussed in Remark 18. Constraint softening allows us to leverage the following properties.

4.3 Nonlinear Programming Properties

In this section, we define NLP properties for the generic problem:

$$\min_{\mathbf{y}} \ \Phi(\mathbf{y}, p) \tag{12a}$$

$$\text{s.t. } c(\mathbf{y}, p) = 0 \tag{12b}$$

$$h(\mathbf{y}, p) \leq 0, \tag{12c}$$

where p is a parameter. These properties are key to support Assumption 16.

Definition 13 *The Lagrange function of* (12) *is:*

$$L(\mathbf{y}) = \Phi(\mathbf{y}, p) + \nu^T c(\mathbf{y}, p) + \eta^T h(\mathbf{y}, p) \tag{13}$$

where ν and η are multipliers of appropriate dimension. A point $\mathbf{y}^*$ is called a KKT-point if there exist multipliers λ and η which satisfy

$$\nabla_{\mathbf{y}} L(\mathbf{y}^*, \nu, \eta, p) \tag{14a}$$

$$c(\mathbf{y}^*, p) = 0 \tag{14b}$$

$$h(\mathbf{y}, p) \leq 0 \tag{14c}$$

$$\eta^T h(\mathbf{y}, p) = 0 \tag{14d}$$

$$\eta \geq 0 \tag{14e}$$

The set of all multipliers ν and η which satisfy the KKT conditions (14) *for a parameter p is $\mathcal{M}(p)$.*

Definition 14 *For problem* (12)*, the Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at the optimal point $\mathbf{y}^*$ if and only if (see [24]):*

- *The rows of $\nabla c(\mathbf{y}^*, p)$ are linearly independent.*
- *There exists a vector $\mathbf{q}$ such that*

$$\nabla_{\mathbf{y}} c(\mathbf{y}^*, p)^T \mathbf{q} = 0, \nabla_{\mathbf{y}} h_j(\mathbf{y}^*, p)^T \mathbf{q} < 0 \quad \text{for all } j \in J \text{ where } J = \{j \mid h_j(\mathbf{y}^*, p) = 0\} \quad (15)$$

As shown in [7], the MFCQ implies that the set of Lagrange multipliers $\mathcal{M}(p)$ remains bounded in a polyhedron.

Definition 15 *The General Strong Second Order Sufficient Condition (GSSOSC) holds at a KKT point $\mathbf{y}^*$ if*

$$\mathbf{q}^T \nabla_{\mathbf{yy}} L(\mathbf{y}^*, \nu, \eta, p) \mathbf{q} > 0 \;\; \forall\, \mathbf{q} \neq 0 \quad (16)$$

such that

$$\nabla_{\mathbf{y}} c(\mathbf{y}^*, p)^T \mathbf{q} = 0, \nabla_{\mathbf{y}} h_j(\mathbf{y}^*, p)^T \mathbf{q} = 0 \quad (17)$$

$$\text{for all } j \in J \text{ where } J = \{j \mid h_j(\mathbf{y}^*, p) = 0, \eta_j > 0\}$$

holds for all $\nu, \eta \in \mathcal{M}(p)$.

5 Properties of eMPC-sc

In this section we discuss properties of the `eMPC-sc` formulation, leading up to our main robustness result. In particular, we will prove that the value function $V(k, \mathbf{w}_k, x_0)$ generated by the `eMPC-sc` controller is a Lyapunov function satisfying the properties of Theorem 8 and thus the system under `eMPC-sc` is ISpS.

Assumption 16 *(A) The set $\mathbb{W}$ is bounded (i.e., $||\mathbf{w}||$ is bounded) and contains zero in its interior. (B) There exists a solution to (11) with $z_N = 0 \;\forall\;\; x_0 \in \mathcal{X}$, $\mathbf{w} \in \mathcal{W}$, $k \in \mathbb{Z}_+$ if $w_{k-1} = 0$. (C) The penalty parameter ρ of (11) is chosen sufficiently large so that $\xi^{ss,L}$, $\xi^{ss,U}$, and ξ^S are zero at the solution of* (11) *$\forall\;\; x_0 \in \mathcal{X}$, $\mathbf{w} \in \mathcal{W}$, $k \in \mathbb{Z}_+$ if $w_{k-1} = 0$. (D) The functions $g_x(\cdot)$, $g_u(\cdot)$, $L^{tr}(\cdot, \cdot)$, $L^{ec}(\cdot, \cdot)$, and $f(\cdot, \cdot, \cdot)$ are twice continuously differentiable. (E) The set $\mathbb{U}$ is convex and compact. (F) The GSSOSC is always satisfied at the solution of* (11)*. (G) There exists $\alpha_1 \in \mathcal{K}_\infty$ such that $L^{tr}(x, u) \geq \alpha_1(|x|)$ for all $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{U}^{n_u}$*

Note that Assumption 16G cannot be relaxed to strict dissipativity since using a rotated version of L^{tr} in (11g) is not necessarily equivalent, in contrast to the regularized case where stage costs are always considered in the objective and optimizing a dissipative stage cost is equivalent to optimizing its rotated counterpart.

Lemma 17 *Under Assumption 16, there exists a solution to (11) with $\xi^S = 0$ for all $k = 1 \dots K-1$, every $x_0 \in \mathcal{X}$, and every admissible $\mathbf{w}_k \in \mathcal{W}$ if $w_{k-1} = 0$.*

Proof. Consider the solution to (11) computed at $k-1$, $\{z_0^{k-1}, z_1^{k-1}, \dots, z_N^{k-1}, v_0^{k-1}, v_1^{k-1}, \dots, v_{N-1}^{k-1}\}$, and assume $w_{k-1} = 0$. Then the shifted solution $\{z_1^{k-1}, z_2^{k-1}, \dots, z_N^{k-1}, 0, v_1^{k-1}, v_2^{k-1}, \dots, v_{N-1}^{k-1}, 0\}$ provides a feasible solution to (11) solved at time k, since $x_k = z_1^{k-1}$. Furthermore, this solution is feasible with $\xi^S = 0$ since $L^{tr}(z_1^{k-1}, v_1^{k-1}) + L^{tr}(z_2^{k-1}, v_2^{k-1}) + \dots + L^{tr}(z_{N-1}^{k-1}, v_{N-1}^{k-1}) - \sum_{i \in \mathcal{N}} L^{tr}(z_i^{k-1}, v_i^{k-1}) = -L^{tr}(x_{k-1}, u_{k-1})$. □

Remark 18 *Note that a penalty parameter ρ exists to satisfy Assumption 16 C when the MFCQ and GSSOSC are satisfied (as discussed in the next section). Choosing ρ larger than the appropriate norm of the multipliers of the solution of problem (9) is sufficient to satisfy Assumption 16 C. Whether Assumption 16C holds can be assessed through off-line simulation of* (11)*, which includes checking for finite-time reachability of x_{ss}.*

Remark 19 *$\xi^S = 0$ is only required in the nominal case. In the case with uncertainty, ξ^S relates to $\sigma(|w_k|)$ in (2b); that is, $\sigma(|w_k|)$ gives an upper bound to ξ^S. Thus, ξ^S may be nonzero, but is bounded because $\sigma(|w_k|)$ and $||\mathbf{w}||$ are bounded.*

Remark 20 *Note that $x_k \in \mathbb{X} \;\forall\; k = 1 \dots K$ is only guaranteed if $x_0 \in \mathbb{X}$, $w_k = 0 \;\forall\; k = 0 \dots K-1$, a solution always exists to (9), and ρ is chosen large enough to force the solutions to (9) and (11) to be equivalent.*

In order to show that the controller is ISpS, we first show that the Lyapunov function is uniformly continuous with respect to disturbances. To this end, we recognize that MPC is a parametric programming problem with respect to x_0. We then make use of Lemma 21 which is a consequence of Theorem 3.1 in [30].

Lemma 21 *If* (11) *satisfies Assumption 16 and the MFCQ, there exists $\sigma_V \in \mathcal{K}$ such that*

$$|V(k, \mathbf{w}_k, x_0) - V(k, (w_0, \dots, w_{k-2}, 0), x_0)| \leq \sigma_V(|w_{k-1}|)$$

This follows because w_{k-1} simply leads to a perturbation of x_0, and it is assumed that f is uniformly continuous in w. We now show that the MFCQ holds.

Lemma 22 *Under Assumption 16, the NLP (11) satisfies the MFCQ.*

Proof. Consider the constraints:

$$z_0 = x_k \tag{18a}$$
$$z_{i+1} = f(z_i, v_i, 0) \quad i = 0, \ldots, N-1 \tag{18b}$$
$$g_u(v_i) \leq 0 \quad i = 0, \ldots, N-1 \tag{18c}$$
$$g_x(z_i) \leq 0 \quad i = 0, \ldots, N \tag{18d}$$
$$g(z_0, v_0, z_1.v_1, \ldots, z_{N-1}, v_{N-1}, z_N) \leq 0 \tag{18e}$$

Linearizing the equality constraints and the *active* inequality constraints of (18) at the solution leads to:

$$F_z d_z + F_v d_v = 0, \quad G^J_{x,z} d_z \leq 0 \tag{19a}$$
$$G^J_{u,v} d_v \leq 0, \quad G^J_z d_z + G^J_v d_v \leq 0 \tag{19b}$$

where d_z and d_v are search directions in the states and controls, respectively, and we define the matrices:

$$F_z = \begin{bmatrix} I & & & & \\ -F^0_z & I & & & \\ & -F^1_z & I & & \\ & & \ddots & \ddots & \\ & & & -F^{N-1}_z & I \end{bmatrix} \tag{20a}$$

$$F_v = \begin{bmatrix} 0 & & & \\ -F^0_v & & & \\ & -F^1_v & & \\ & & \ddots & \\ & & & -F^{N-1}_v \end{bmatrix} \tag{20b}$$

$$G^J_{x,z} = diag\{G^{j_0}_{x,z}, G^{j_1}_{x,z}, \ldots, G^{j_{N-1}}_{x,z}, G^{j_N}_{x,z}\} \tag{20c}$$
$$G^J_{u,v} = diag\{G^{j_0}_{u,v}, G^{j_1}_{u,v}, \ldots, G^{j_{N-1}}_{u,v}, 0\} \tag{20d}$$
$$G^J_z = \begin{bmatrix} G^{j_0}_z & G^{j_1}_z & \ldots & G^{j_{N-1}}_z & G^{j_N}_z \end{bmatrix} \tag{20e}$$
$$G^J_v = \begin{bmatrix} G^{j_0}_v & G^{j_1}_v & \ldots & G^{j_{N-1}}_v & 0 \end{bmatrix} \tag{20f}$$

where F^i_z and F^i_v are the Jacobians of $f(z_i, v_i, 0)$ with respect to variables z and v, $G^{J_i}_{x,z}$ is the Jacobian of the active constraints of g_x at time i, $G^{J_i}_{u,v}$ is the Jacobian of the active constraints of g_u at time i, and $G^{J_i}_z$ and $G^{J_i}_v$ are the Jacobians of g with respect to z and v. We see that F_z is square and nonsingular and that the matrix $\nabla c^T = [F_z \mid F_v \mid 0]$ is full row rank. Hence, the equality constraint gradients are linearly independent. Also, the submatrices $G^{j_i}_z$ and $G^{j_i}_v$ may be of variable dimension and even be empty. Furthermore, the relaxation of the active state constraints of (18) at the optimum leads to:

$$G^J_{x,z} d_z - E_{J,x} d_{\xi,x} \leq 0, \quad d_{\xi,x} \geq 0 \tag{21a}$$
$$G^J_{u,v} d_v \leq 0 \tag{21b}$$
$$G^J_z d_z + G^J_v d_v - E_J d_\xi \leq 0, \quad d_\xi \geq 0 \tag{21c}$$

where ξ is the concatenation of the ℓ_1 penalty relaxation variables and

$$\nabla g^T_J = \begin{bmatrix} G^J_z & G^J_v & -E_J & 0 \\ 0 & 0 & -I & 0 \\ G^J_{x,z} & 0 & 0 & -E_{J,x} \\ 0 & 0 & 0 & -I \\ 0 & G^J_{u,v} & 0 & 0 \end{bmatrix} \tag{22}$$

where E_J is composed of the rows of the identity matrix that correspond to the active inequalities.

Since the set $\mathbb{U}$ is convex and has an interior, we can find some d^0_v such that $g^j_u(v^* + d^0_v) < 0 \; \forall j \in J$. Applying Taylor's Theorem gives:

$$g^j_u(v^* + d^0_v) = g^j_u(v^*) + \nabla g^j(v^*)^T d^0_v + \frac{1}{2}(d^0_v)^T \nabla^2 g^j_u(v^* + t d^0_v) d^0_v < 0 \tag{23}$$

for $j \in J$ and some $t \in [0, 1]$. Since $g^j_u(v^*) = 0$ and by convexity $\frac{1}{2}(d^0_v)^T \nabla^2 g^j_u(v^* + t d^0_v) d^0_v \geq 0$, we have

$$\nabla g^j_u(v^*)^T d^0_v = g^j_u(v^* + d^0_v) - \frac{1}{2}(d^0_v)^T \nabla^2 g^j_u(v^* + t d^0_v) d^0_v < 0 \tag{24}$$

so we have $G^J_{u,v} d^0_v < 0$.

Now define the concatenated variables and set $q^T = [d^T_z \mid d^T_v \mid d^T_\xi \mid d^T_{\xi,x}]$ with $d_z = -F^{-1}_z F_v d^0_v$, $d_v = d^0_v$. Then given d^0_v choose d_ξ and $d_{\xi,x}$ as follows:

$$E_J d_\xi > (G^J_v - G^J_z F^{-1}_z F_v) d^0_v \tag{25a}$$
$$E_{J,x} d_{\xi,x} > -G^J_{x,z} F^{-1}_z F_v d^0_v \tag{25b}$$

and we see that $\nabla g^T_J q < 0$ and $\nabla c^T q = 0$ in Definition (14). Hence MFCQ is satisfied for system (18). □

Remark 23 *GSSOSC requires that the solution of the NLP satisfies the strong second order conditions for every Lagrange multiplier in the bounded set defined under MFCQ. If this does not hold, regularization to add positive curvature will be required. This can always be enforced by adding* $\|\mathbf{y} - \mathbf{y}^*\|^2_Y$ *to the objective of* (12) *after the KKT point* $\mathbf{y}^*$ *is found, with matrix* Y *sufficiently positive definite [38]. This ensures that* (11) *satisfies GSSOSC and does not change the solution* $\mathbf{y}^*$*. Note* *that the stabilizing constraint of* `eMPC-sc` *can help induce positive curvature if the tracking stage cost is quadratic.*

5.1 ISpS of `eMPC-sc`

We now prove our main result: that the dynamic system under the `eMPC-sc` controller is ISpS.

Theorem 24 *If Assumption 16 holds, there exists* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, *and* $c_1, c_2 \in \mathbb{R}_+$ *such that* $V(k, \mathbf{w}_k, x_0)$ *satisfies* (2), *and the system under* `eMPC-sc` *is ISpS with respect to* $c = \alpha_c(\hat{u})$, *for some* $\alpha_c \in \mathcal{K}_\infty$ *for all* $x_0 \in \mathcal{X}, \mathbf{w} \in \mathcal{W}, k \in \mathbb{Z}_+$.

Proof. We assume that L^{tr} has a $\mathcal{K}_\infty$ lower bound, so that $\alpha_1(|x|) \leq L^{tr}(x,u)$, which gives a lower bound for the sum. Second, since we assume that (11g) holds with $\xi^S = 0$ in the nominal case, we have that $\alpha_3(|x|) = \delta\, \alpha_1(|x|)$ in (2). To show the upper bound of V, we must assume uniform continuity of the stage cost $L^{tr}(\cdot,\cdot)$ and of the nonlinear system $f(\cdot)$. This ensures that $L^{tr}(x,u) \leq \sigma_L(|x| + |u|)$ and $f(x,u,0) \leq \sigma_f(|x| + |u|)$ hold for $\sigma_L(\cdot), \sigma_f(\cdot) \in \mathcal{K}$. This allows us to establish an upper bound of the constrained sum from (11g) of the form:

$$\sum_{i\in\mathcal{N}} L^{tr}(z_i, v_i)$$

$$\leq \sigma_L(|x_k| + \hat{u}) + \sum_{i=1}^{N-1} L^{tr}(z_i, v_i) \tag{26a}$$

$$\leq \sigma_L(|x_k| + \hat{u}) + \sigma_L(\sigma_f(|x_k| + \hat{u}) + \hat{u}) \tag{26b}$$

$$+ \sum_{i=2}^{N-1} L^{tr}(z_i, v_i) \tag{26c}$$

$$\leq \sigma_L(|x_k| + \hat{u}) + \sigma_L(\sigma_f(|x_k| + \hat{u}) + \hat{u}) + \sigma_L(\sigma_f(\sigma_f(|x_k| + \hat{u}) + \hat{u}) + \hat{u}) + \sum_{i=3}^{N-1} L^{tr}(z_i, v_i) \tag{26d}$$

$$=: \alpha(|x_k| + \hat{u}) \tag{26e}$$

for some $\alpha \in \mathcal{K}_\infty$. We then have that (2a) holds with $\alpha_2(|x_k|) := \alpha(|x_k| + \hat{u}) - \alpha(\hat{u})$ and $c_1 := \alpha(\hat{u})$.

Next, by Assumption 16C, ρ is chosen large enough such that (11g) satisfies $\xi^S = 0$ in the nominal case. Then, since (11) satisfies MFCQ by Lemma 22, and by Assumption 16F the GSSOSC holds, we have from Lemma 21:

$$\begin{aligned}
&V(k+1, \mathbf{w}_{k+1}, x_0) - V(k, \mathbf{w}_k, x_0) = \\
&V(k+1, \mathbf{w}_{k+1}, x_0) - V(k+1, (w_0, \ldots, w_{k-1}, 0), x_0) \\
&+V(k+1, (w_0, \ldots, w_{k-1}, 0), x_0) - V(k, \mathbf{w}_k, x_0) \\
&\leq -\alpha_3(|x_k|) + \sigma_V(|w_k|)
\end{aligned}$$

Consequently, (2) is satisfied and the system is ISpS with $c_1 = \alpha(\hat{u})$, where $\alpha \in \mathcal{K}_\infty$, and $c_2 = 0$. Furthermore, from the proof of Theorem 8 we have that c is a $\mathcal{K}_\infty$ function of c_1. The proof is complete. □

Remark 25 *It is important to highlight that there exists an upper bound on* c*, used in the definition of ISpS, which depends on the bound* $\hat{u}$*.. From the proof of Theorem 8, it is clear that the constant* c *used in the definition of ISpS is a* $\mathcal{K}_\infty$ *function of* c_1 *which is in turn a* $\mathcal{K}_\infty$ *function of* $\hat{u} = \max_{u\in\mathbb{U}} |u|$*. Consequently,* c *is a* $\mathcal{K}_\infty$ *function of* $\hat{u}$*.*

Remark 26 *We highlight that, the only properties that rely on the assumption that the the penalty parameter* ρ *is larger than the norm of the Lagrange multipliers, are recursive feasibility (Lemma 17) and nominal constraint satisfaction (Remark 20). Otherwise,* $\rho \geq 0$ *is the only requirement on* ρ *for stability properties to hold. In other words, bounds on Lagrange multipliers are not needed for stability.*

5.2 Observations on the Nominal Case

The constant $c_1 = \alpha(\hat{u})$ (and associated c) used in the proof of Theorem 24 play a critical role in the theoretical properties of the `eMPC-sc` controller and is tightly connected to weak controllability (see Assumption 1 in [3]). To see the implications of this, we make the following observations. First note that, since $c_2 = 0$, `eMPC-sc` becomes ISS if $c_1 = 0$ holds (which implies that $c = 0$ holds as well). This is important because, whenever $c_1 = 0$, we have that asymptotic stability holds in the nominal case, and ISS holds in the case with uncertainty. In general, however, we have that $c_1 > 0$ (and thus $c > 0$) holds and thus `eMPC-sc` is only ISpS and not necessarily asymptotically stable in the nominal case.

The upper bound condition of the cost function (26d) directly relates the constant c_1 to violation of weak controllability. In particular, $c_1 > 0$ implies that the implicit control law $\mathbf{u}(x_k)$ may not satisfy $\mathbf{u}(0) = 0$, as is required for weak controllability, which can be advantageous in terms of economic performance. If controller (11), satisfies Definition 12 with $\omega = 0$ (a non-typical case), then weak controllability can be obtained, since uniform continuity of the solution also holds. Otherwise, we can still show the following nominal result.

Assumption 27 *(A)* $w_k = 0 \ \ \forall\, k \in \mathbb{Z}_+$ *(B) There exists a solution to* (11) *with* $z_N = 0 \;\forall\; x_0 \in \mathcal{X}$*. (C) The penalty parameter* ρ *of* (11) *is chosen sufficiently large so as to force* $\xi^{ss,L}$*,* $\xi^{ss,U}$*, and* ξ^S *to zero at the solution of* (11) $\forall\ \ k \in \mathbb{Z}_+$ *(C) The functions* $g_x(\cdot)$*,* $g_u(\cdot)$*,* $L^{tr}(\cdot,\cdot)$*,* $L^{ec}(\cdot,\cdot)$*, and* $f(\cdot,\cdot,\cdot)$ *are twice continuously differentiable. (E) The set* $\mathbb{U}$ *is convex and compact. (D) There exists* $\alpha_1 \in \mathcal{K}_\infty$ *such that* $L^{tr}(x,u) \geq \alpha_1(|x|)$ *for all* $x \in \mathbb{R}^{n_x}$*,* $u \in \mathbb{U}^{n_u}$

Theorem 28 *If Assumption 27 holds, then the system under eMPC-sc is attractive for all* $x_0 \in \mathcal{X}$*.*

Proof. Take the infinite sum of (2b) with $\sigma(|w_k|) = 0$ (true with no uncertainty) and $c_2 = 0$ (true for eMPC-sc in general):

$$\sum_{k=0}^{\infty} \alpha_3(x_k) \leq \sum_{k=0}^{\infty} (V(k, \mathbf{w}_k, x_0) - V(k+1, \mathbf{w}_{k+1}, x_0)) \tag{27a}$$

$$= V(0, \mathbf{w}_0, x_0) \leq \alpha_2(|x_0|) + c_1. \tag{27b}$$

□

Remark 29 *In this work we have exclusively used an endpoint equality constraint. This terminal constraint is only important to show recursive feasibility in Lemma 17. To that end, another terminal cost and/or terminal region could also be employed, but we use the endpoint constraint here for simplicity.*

6 Case Studies

6.1 Nonlinear CSTR

For our first example we consider a nonlinear continuously stirred tank reactor (CSTR) from [3]:

$$\frac{dc_A}{dt} = \frac{q}{V}(c_{Af} - c_A) - kc_A \tag{28a}$$

$$\frac{dc_B}{dt} = \frac{q}{V}(-c_B) + kc_A \tag{28b}$$

where c_A and c_B denote the concentrations of components A and B, respectively, in mol/l. The manipulated input is q in l/min, the reactor volume is $V = 10\ l$, the rate constant is $k = 1.2\ l/(mol{\cdot}min)$, and $c_{Af} = 1\ mol/l$ is the feed concentration. In addition, we set variable bounds as $10 \leq q \leq 20$ and $0.45 \leq c_B \leq 1$ which are softened with slack variables that are penalized in the objective function in the NLP. The economic stage cost is:

$$L^{ec}(c_A, c_B, q) = -q\left(2c_B - \frac{1}{2}\right) \tag{29}$$

The steady state used for the tracking objective is $c_A^* = 0.5$, $c_B^* = 0.5$, $q^* = 12$, so $L_{ss}^{ec} = -6$. We compare the performance of `eMPC-sc` with a regularized economic MPC controller. Regularization weights for the latter are calculated in [36] and are $w_A = 1.01$, $w_B = 0.01$, and $w_q = 1.01$ (assuming $\omega = 1$), so the tracking stage cost is:

$$L^{tr}(c_A, c_B, q) = 1.01c_A^2 + .5c_B^2 + 12q^2 \tag{30}$$

We discretize the system using three point Radau collocation and a finite element length of 1 min. The penalty parameter is chosen as $\rho = 10^8$. We use condition $c_A^0 = .1$, $c_B^0 = 1$. We implement the problem in AMPL [6] and solve the NLPs with IPOPT [35].

First we consider the effect of δ in (11g) on the performance of `eMPC-sc`, using a horizon of $N = 100$. To investigate this, we first simulate forward for 5 time steps, then solve the NLP at the resulting initial condition. This case is repeated for several δ values. The existence of an economically optimal periodic orbit causes eMPC-sc to become oscillitory, as shown in Figures 1 and 2.Results for this are shown in Table 1. The sum of predicted costs shows a monotonic relationship with δ.

Table 1
CSTR example, comparison of solutions with varying δ, constant $N = 100$ and x_k

δ	$\sum_{i\in\mathcal{N}} L^{ec}(z_i, v_i) - L_{ss}^{ec}$
.99	-28.0709
.9	-28.4348
.7	-28.461
.5	-28.4872
.3	-28.5134
.1	-28.5396
.01	-28.5514

Economic results for the case without uncertainty are shown in the left column of Table 2. Economic results for the case with additive state uncertainty w_k (with standard deviation of 0.01 and zero mean) are shown in the right column of Table 2. We use the same realization of the uncertainty in every case. Both with and without uncertainty, we consider the tracking case (3), the economic case (7), the regularization case $\texttt{eMPC} - \texttt{reg}$ (8) with different fractions of the sufficient weight, and `eMPC-sc` (11) with different values of δ (inequality constraints are softened in each formulation with uncertainty). A short simulation length of $K = 9$ is chosen so that we can focus on the dynamics that occur before the set-point is reached.

Similar trends are seen both with and without uncertainty. The cases with regularization (8) have similar cost to the tracking case (3), and the cases with a stabilizing constraint (11) have similar cost to the economic case (7). Costs roughly decrease with the regularization weight and with δ, although this relationship is not perfectly monotonic. Even though we expect that total predicted costs to be monotonic with δ, the sum of implemented costs does not need to be, due to potential local solutions and a finite horizon in (11).

Finally we examine the dynamic behaviors for select cases. We show the eMPC-sc case with $\delta = 0.99$, with and without uncertainty in Figures 1 and 2, respectively. We include two different horizon lengths. We note that this system has the property that maximum economic performance is obtained under a periodic orbit [3], which

Table 2
CSTR example with $N = 100$, comparing the accumulated cost $\sum_{k=0}^{K} L^{ec}(x_k, u_k) - L^{ec}_{ss}$

Case	w/o uncertainty	w/ uncertainty
Tracking	-14.0177	-12.689
`eMPC-reg` 100 %	-14.3527	-13.0301
75%	-14.4694	-13.1488
50%	-14.7109	-13.3944
25%	-15.5087	-14.2052
Average	-14.7604	-13.4446
`eMPC-sc` $\delta = 0.99$	-20.8853	-19.3943
0 .9	-20.8774	-19.3459
0 .7	-20.9073	-19.3292
0.5	-20.9122	-19.3692
0.3	-20.9106	-19.3529
0.1	-20.9116	-19.389
0.01	-20.9168	-19.4038
Average	-20.9030	-19.3692
Purely economic	-21.7259	-20.2366

explains the oscillatory behavior. The proposed eMPC-sc controller allows the system to explore that orbit to gain economic performance but the stabilizing constraint eventually forces it to converge to the equilibrium point. We note that the system converges to the equilibrium point in the nominal case and to a neighborhood of the equilibrium point in the robust case. However, the speed and manner of convergence is not guaranteed. Note that convergence is much faster for the shorter horizon, as (11g) is more constraining in this case. Also, for the case where $\delta = 0.01$ (not shown in figures), convergence takes about 12,000 time steps.

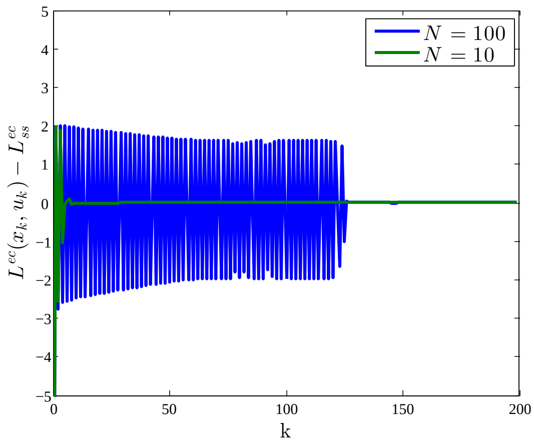

Fig. 1. CSTR, eMPC-sc with $\delta = 0.99$, no uncertainty

The stabilizing constraint used in `eMPC-sc` has demonstrated a clear economic benefit for this example. Again, note that this constraint is easy to implement, unlike the regularization weights, which require offline calculations and are overly conservative. Also note that the economic performance is generally poorer with uncertainty, which commonly occurs in eMPC. Nevertheless, the `eMPC-sc` provides clear benefits here as well.

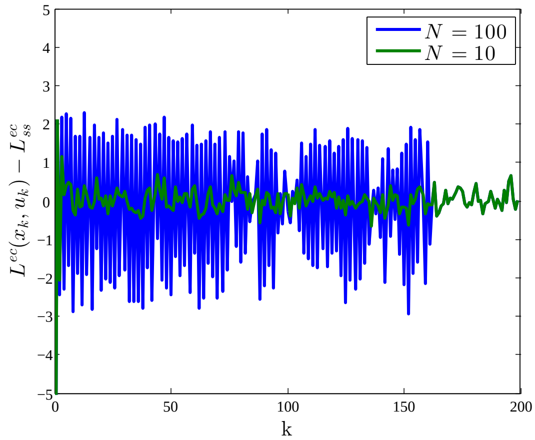

Fig. 2. CSTR, eMPC-sc with $\delta = 0.99$ and uncertainty

6.2 *Large-Scale Distillation System*

We consider the challenging nonlinear system shown in Figure 3 with two distillation columns in series [20]. Each column is based on the model described in [31], with the main difference that we consider three components, A, B, and C. The bottoms of the first column are the feed to the second column. The flowsheet is shown in Figure 3. The distillate of the first column is to be 95% A, the distillate of the second column is to be 95% B, and the bottoms of the second column is to be 95% C. The thermodynamics assume constant relative volatility, and for tray hydraulics, we use the Francis weir formula, with constant $K_{uf} = 21.65$ above the feed and constant $K_{bf} = 29.65$ below the feed. The weir height is 0.25, and the nominal liquid holdup is 0.5. Each column has 41 equilibrium stages including the reboiler, giving a total of 246 states and 8 controls. The model for each column is given below, with variable and parameter definitions in Table 3. Note that the subscript denoting which column a variable pertains to is ignored in (31)-(38) and Table 3 for concision.

Vapor Liquid Equilibria

$$a_{i,j} = x_{i,j}\alpha_j \quad \forall\, i \in \{0, .., NT-1\}, j \in \{A, B\} \tag{31a}$$

$$b_i = x_{i,1}(\alpha_A - 1) + x_{i,2}(\alpha_B - 1) + 1 \quad \forall\, i \in \{0, .., NT-1\} \tag{31b}$$

$$y_{i,j} * b_i = a_{i,j} \quad \forall\, i \in \{0, .., NT-1\}, j \in \{A, B\} \tag{31c}$$

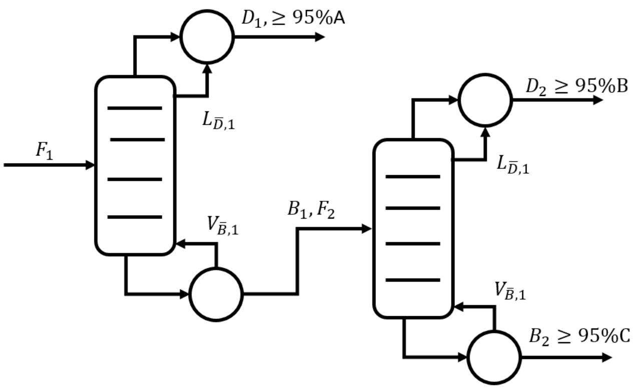

Fig. 3. Distillation Flowsheet

Vapor Flows

$$V_i = V_{\bar{B}} \quad \forall\, i \in 1..NF-1 \tag{32a}$$

$$V_i = V_{\bar{B}} + (1-q_F)F \quad \forall\, i \in NF..NT-1 \tag{32b}$$

Liquid Flows

$$L_i = K_{bf}\left(\frac{(M_i - M_{uw}) + \sqrt{(M_i - M_{uw})^2}}{2}\right)^{1.5}$$

$$\forall\, i \in \{2,..,NF\} \tag{33a}$$

$$L_i = K_{uf}\left(\frac{(M_i - M_{uw}) + \sqrt{(M_i - M_{uw})^2}}{2}\right)^{1.5}$$

$$\forall\, i \in \{NF+1,..,NT-1\} \tag{33b}$$

$$L_{NT} = L_{\bar{D}} \tag{33c}$$

Holdup Balances

$$dM_i/dt = L_{i+1} - L_i + V_{i-1} - V_i$$

$$\forall i \in \{2,..,NT-1\}/\{NF\} \tag{34a}$$

$$d(M_i x_{i,j})/dt = L_{i+1}x_{i+1,j} - L_i x_{i,j} + V_{i-1}y_{i-1,j}$$

$$-V_i y_{i,j} \quad \forall\, i \in \{2,..,NT-1\}/\{NF\}, j \in \{A,B\} \tag{34b}$$

Feed Balance

$$dM_{NF}/dt = L_{NF+1} - L_{NF} + V_{NF-1} - V_{NF} + F$$

$$\forall\, j \in \{A,B\} \tag{35a}$$

$$d(M_{NF}x_{NF,j})/dt = L_{NF+1}x_{NF+1,j} - L_{NF}x_{NF,j}$$

$$+V_{NF-1}y_{NF-1,j} - V_{NF}y_{NF,j} + Fz_{F\,j} \quad \forall\, j \in \{A,B\} \tag{35b}$$

Reboiler Balance (equilibrium stage)

$$dM_1/dt = L_2 - V_{\bar{B}} - \bar{B} \tag{36a}$$

$$d(M_1 x_{1,j})/dt = L_2 x_{2,j} - V_{\bar{B}} y_{1,j} - \bar{B}x_{1,j} \quad \forall\, j \in \{A,B\} \tag{36b}$$

Condenser Balance

$$dM_{NT}/dt = V_{NT-1} - L_{\bar{D}} - \bar{D} \tag{37a}$$

$$d(M_{NT}x_{NT,j})/dt = V_{NT-1}y_{NT-1,j} - L_{\bar{D}}x_{NT,j}$$

$$-\bar{D}x_{NT,j} \quad \forall\, j \in \{A,B\} \tag{37b}$$

Concentrations

$$M_i\, dx_{i,j}/dt = d(M_i x_{i,j})/dt - x_{i,j}\; dM_i/dt$$

$$\forall i \in \{1,..,NT\}, j \in \{A,B\} \tag{38}$$

Table 3
Distillation model definitions, i is indexed over trays, j is indexed over components

Variable/Parameter	Definition
$a_{i,j}$,b_i	thermodynamic variables
$y_{i,j}$	vapor mol fraction
$\alpha_{i,j}$	thermodynamic parameters
V_i	vapor flow
$V_{\bar{B}}$	vapor boilup
NT	number of trays
NF	tray number of feed
M_{uw}	weir height
F	feed flow
L_i	liquid flow
q_F	fraction of lquid in feed
$L_{\bar{D}}$	reflux flow
$x_{i,j}$	liquid mol fraction
M_i	tray holdup
z_F	feed mol fraction
$\bar{D}$	distillate flow
$\bar{B}$	bottoms flow
K_{uf}, K_{bf}	weir constants

We set $\alpha_A = 2$ and $\alpha_B = 1.5$. The economic cost is the cost of feed and energy to the reboilers minus the cost of the products, that is $L^{ec} = p_F \cdot F_1 + pV(V_{\bar{B},1} + V_{\bar{B},2}) - p_A \cdot D_1 - p_B \cdot D_2 - p_C \cdot B_2$, where $p_F = \$1/mol$ is the price of feed, p_i for $i = A, B, C$ is the price of component i with $p_A = \$1/mol$, $p_B = \$2/mol$, and $p_C = \$1/mol$, $p_V = \$0.008/mol$ is the price per mole vaporized in the reboilers, and the indices represent the first or second column. The feed is saturated liquid, and the composition of the feed is 0.4 mole fraction A, 0.2 mole fraction B, and 0.4 mole fraction C. The purities are implemented as inequality constraints. We discretize the DAE system using three point Radau collocation, and we use a finite element length of 1 min and $N = 25$. Each NLP (11) has $120,000$ variables, $108,000$ equality constraints, and $14,000$ inequality constraints. The models are implemented in AMPL and solved with IPOPT.

Finding sufficient regularization weights in this case is much more cumbersome due to the size of the system. Here, the Hessian of the steady state problem must be found at many points in the state space. We refer to the regularized case using weights obtained from the Gershgorin Circle theorem as the 100% regularization instance and we relaxed this instance by using smaller percentages. The Gershgorin weights are reported and compared in [36]. We highlight that, in order to rigorously enforce strict dissipativity, the Hessian must be checked at every possible point of operation in the state space, a difficult task for a problem of this size and complexity. We also define L^{tr} such that these weights are on the diagonals of matrices Q and R and $L^{tr}(x, u) = x^T Qx + u^T Ru$.

We first compare solution times for the specific cases that `eMPC-sc` (11) is implemented with $\delta = 0.01$ and of `eMPC-reg` (8) with 100% of the Gershgorin weight. The `eMPC-sc` problems can be solved in approximately 271 seconds and 188 IPOPT iterations, while `eMPC-reg` averages 83 seconds and 70 iterations. This clearly illustrates that regularization is beneficial for computational performance but sacrifices economic performance.

A comparison of accumulated stage costs over ten time steps ($\sum_{k=0}^{K} L^{ec}(x_k, u_k) - L^{ec}_{ss}$) from the same initial condition for various cases is shown in the left column of Table 4. Again, we choose a short simulation length of $K = 9$ to emphasize dynamic performance. The penalty parameter is chosen as $\rho = 10^4$. The steady state cost, L^{ec}_{ss}, is -0.223.

From the nominal results, it is apparent that `eMPC-sc` provides economic benefit over regularized fomulation `eMPC-reg`, but the accumulated cost is again not monotonic with δ, due to a short horizon and local solutions. It is also interesting that reducing the regularization weight (by a given percentage) does little to improve performance.

We also consider the cases with uncertainty w_k in the feed flow rate and composition in (35). We use additive uncertainty with values of w_k sampled from a normal distribution. The disturbance w_k has zero mean and standard deviation 0.1 for the feed flow rate and 0.01 for the mole fractions of A and B. These results are shown in the right column of Table 4, and `eMPC-sc` (11) again shows a clearly improved economic performance over `eMPC-reg`

Table 4
Distillation example, $\sum_{k=0}^{K} (L^{ec}(x_k, u_k) - L^{ec}_{ss})$

Case	nominal	w/ uncertainty
Tracking	-20.903	-20.736
`eMPC-reg` 100 %	-22.665	-21.662
75%	-22.675	-21.599
50%	-22.676	-21.511
25%	-22.658	-21.361
Average	-22.668	-21.534
`eMPC-sc`, $\delta = 0.99$	-25.876	-24.283
0.9	-28.933	-24.162
0.7	-28.406	-24.947
0.5	-29.701	-24.234
0.3	-27.481	-25.199
0.1	-29.453	-25.656
0.01	-29.693	-24.039
Average	-28.506	-24.646
Economic	-27.081	-24.479

(8). However, economic performance is decreased due to the presence of disturbances.

7 Conclusions and Future Work

We have established robustness properties for an economic MPC controller that trades off convergence rate and economic performance, but does not require strict dissipativity with respect to stage costs used in the objective. In particular, we show that this controller is input-to-state practically stable under reasonable assumptions. Computational studies show significantly improved economic performance compared to existing regularization approaches. Future work will explore further extensions to `eMPC-sc`, primarily focused on the relaxation of the terminal constraint in Assumption 16 B through the construction of a terminal cost, and finding a way to tune c_1 via including a constraint on the controls u in (11). Also, a stabilizing constraint applied to cyclic steady states [15] may be considered.

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8 Appendix

In this appendix we present a proof for Theorem 8, which states that if the system admits a modified Lyapunov function then it satisfies ISpS. We begin by stating the following preliminary results.

Lemma 30 *For a function $V(k, \mathbf{w}_k, x_0)$ satisfying (2), there exists $\alpha_4(\cdot) \in \mathcal{K}_\infty$ such that*

$$\alpha_3(|x_k|) \geq \alpha_4(V(k, \mathbf{w}_k, x_0)) - \bar{c} \quad \forall\ x_0 \in \mathcal{X},\ \mathbf{w} \in \mathcal{W},\ k \in \mathbb{Z}_+ \tag{39}$$

for some $\bar{c} \in \mathbb{R}_+$.

Proof. Given $V(k, \mathbf{w}_k, x_0)$ and its upper bound, we can find a lower bound of $|x_k|$:

$$|x_k| \geq \begin{cases} \alpha_2^{-1}(V(k, \mathbf{w}_k, x_0) - c_1) \\ \qquad \forall\ V(k, \mathbf{w}_k, x_0) \geq c_1 \\ 0 \ \ \forall\ V(k, \mathbf{w}_k, x_0) < c_1 \end{cases} \tag{40a}$$

$$\Rightarrow \alpha_3(|x_k|) \geq \begin{cases} \alpha_3 \circ \alpha_2^{-1}(V(k, \mathbf{w}_k, x_0) - c_1) \\ \qquad \forall\ V(k, \mathbf{w}_k, x_0) \geq c_1 \\ 0 \ \ \forall\ V(k, \mathbf{w}_k, x_0) < c_1 \end{cases} \tag{40b}$$

$$\geq \begin{cases} \alpha_3 \circ \alpha_2^{-1}(V(k, \mathbf{w}_k, x_0) - c_1) \\ \qquad \forall\ V(k, \mathbf{w}_k, x_0) \geq c_1 \\ (\alpha_3 \circ \alpha_2^{-1}(c_1)/c_1)V(k, \mathbf{w}_k, x_0) - \alpha_3 \circ \alpha_2^{-1}(c_1) \\ \qquad \forall\ V(k, \mathbf{w}_k, x_0) < c_1 \end{cases} \tag{40c}$$

$$=: \alpha_4(V(k, \mathbf{w}_k, x_0)) - \alpha_3 \circ \alpha_2^{-1}(c_1) \tag{40d}$$

Then set

$$\alpha_4(V(k, \mathbf{w}_k, x_0)) := \begin{cases} \alpha_3 \circ \alpha_2^{-1}(V(k, \mathbf{w}_k, x_0) - c_1) + \alpha_3 \circ \alpha_2^{-1}(c_1) \\ \qquad \forall\ V(k, \mathbf{w}_k, x_0) \geq c_1 \\ (\alpha_3 \circ \alpha_2^{-1}(c_1)/c_1)V(k, \mathbf{w}_k, x_0) \\ \qquad \forall\ V(k, \mathbf{w}_k, x_0) < c_1 \end{cases} \tag{41}$$

and set

$$\bar{c} := \alpha_3 \circ \alpha_2^{-1}(c_1) \tag{42}$$

□

Lemma 31 *For every $\hat{\beta} \in \mathcal{KL}$ and $\hat{c} \in \mathbb{R}_+$, there exists some $\beta \in \mathcal{KL}$ and $\tilde{c} \in \mathbb{R}_+$ such that:*

$$\hat{\beta}(s + \hat{c}, k) \leq \beta(s, k) + \tilde{c} \ \ \forall\ s \in \mathbb{R}_+,\ k \in \mathbb{Z}_+ \tag{43}$$

Proof. Consider $\tilde{c} = \hat{\beta}(\hat{c}, 0)$ and any $\bar{\beta} \in \mathcal{KL}$. Then let:

$$\beta(s, k) = \begin{cases} \hat{\beta}(s + \hat{c}, k) - \tilde{c} + \bar{\beta}(s, k) \\ \qquad \forall\ s \in \mathbb{R}_+,\ k \in \mathbb{Z}_+ \text{ s.t. } \hat{\beta}(s + \hat{c}, k) - \tilde{c} \geq 0 \\ \bar{\beta}(s, k) \\ \qquad \forall\ s \in \mathbb{R}_+,\ k \in \mathbb{Z}_+ \text{ s.t. } \hat{\beta}(s + \hat{c}, k) - \tilde{c} < 0 \end{cases} \tag{44}$$

Then $\beta(s, k)$ clearly gives an upper bound to $\hat{\beta}(s+\hat{c}, k) - \tilde{c}$, but to see that $\beta \in \mathcal{KL}$, consider more closely the expression $\hat{\beta}(s + \hat{c}, k) - \tilde{c}$. For constant k, $\hat{\beta}(s + \hat{c}, k) - \tilde{c}$ is a $\mathcal{K}_\infty$ function minus a positive constant. Thus, the regions such that $\hat{\beta}(s+\hat{c}, k) - \tilde{c} \geq 0$ and $\hat{\beta}(s+\hat{c}, k) - \tilde{c} < 0$ are described by intervals $s \in [r, \infty)$ and $s \in [0, r)$, respectively, for some $r \in \mathbb{R}_+$. Thus $\beta(0, k) = 0$, and $\beta(s, k)$ is continuous and strictly increasing with respect to s. A similar observation holds for constant s. Now, the regions such that $\hat{\beta}(s+\hat{c}, k) - \tilde{c} \geq 0$ and $\hat{\beta}(s+\hat{c}, k) - \tilde{c} < 0$ are described by intervals $k \leq p$ and $k > p$, respectively, for some $p \in \mathbb{Z}_+$. Thus $\beta(s, k)$ is strictly decreasing with respect to k. Therefore, $\beta \in \mathcal{KL}$. □

Proof of Theorem 8. We denote id as the identity function, and the notation $\alpha_1 \circ \alpha_2(\cdot)$ denotes $\alpha_1(\alpha_2(\cdot))$. Define the functions $\rho(\cdot)$ and $\hat{\alpha}_4(\cdot)$ to have the following properties: $\hat{\alpha}_4 \in \mathcal{K}_\infty$, $\hat{\alpha}_4(s) \leq \alpha_4(s) \ \ \forall\ s$, $id - \hat{\alpha}_4(\cdot) \in \mathcal{K}_\infty$, $\rho(\cdot) \in \mathcal{K}_\infty$, and $id - \rho(\cdot) \in \mathcal{K}_\infty$. See Lemma B.1 of [18] for a proof that $\hat{\alpha}_4(\cdot)$ exists.

We now proceed by showing that β, γ, and c exist satisfying ISpS. Our proof follows along the lines of Lemma 3.5 in [18]. Define the constant $b := \hat{\alpha}_4^{-1} \circ \rho^{-1}(\sigma(||\mathbf{w}||) + \bar{c} + c_2)$ with $\bar{c}$ defined in Lemma 30 and consider the set $D = \{x : V(x) \leq b\}$. If $x_k \in \mathcal{X} \cap D$ then we have from (2b), Lemma 30, and the definition of b that:

$$V(k + 1, \mathbf{w}_{k+1}, x_0)$$

$$\leq V(k, \mathbf{w}_k, x_0) - \alpha_3(|x_k|) + \sigma(||\mathbf{w}||) + c_2 \tag{45a}$$

$$\leq V(k, \mathbf{w}_k, x_0) - \alpha_4(V(k, \mathbf{w}_k, x_0)) + \bar{c} + \sigma(||\mathbf{w}||) + c_2 \tag{45b}$$

$$\leq V(k, \mathbf{w}_k, x_0) - \hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) + \bar{c} + \sigma(||\mathbf{w}||) + c_2 \tag{45c}$$

$$= (id - \hat{\alpha}_4) \circ V(k, \mathbf{w}_k, x_0) + \bar{c} + \sigma(||\mathbf{w}||) + c_2 \tag{45d}$$

$$\leq (id - \hat{\alpha}_4) \circ b + \bar{c} + \sigma(||\mathbf{w}||) + c_2 \tag{45e}$$

$$= (id - \hat{\alpha}_4) \circ b + \rho \circ \hat{\alpha}_4(b) \tag{45f}$$

$$= -(id - \rho) \circ \hat{\alpha}_4(b) + b \tag{45g}$$

$$\leq b. \tag{45h}$$

This holds for all $x_k \in D \cap \mathcal{X},\ \mathbf{w} \in \mathcal{W}$. Consequently, the set D is positive invariant. So, set:

$$\gamma(||\mathbf{w}_k||) = \alpha_1^{-1} \circ \hat{\alpha}_4^{-1} \circ \rho^{-1}(\sigma(||\mathbf{w}||) + \bar{c} + c_2) \\ -\alpha_1^{-1} \circ \hat{\alpha}_4^{-1} \circ \rho^{-1}(\bar{c} + c_2) \tag{46}$$

and

$$c_3 = \alpha_1^{-1} \circ \hat{\alpha}_4^{-1} \circ \rho^{-1}(\bar{c} + c_2) \tag{47}$$

so that

$$\gamma(||\mathbf{w}_k||) + c_3 = \alpha_1^{-1}(b) \tag{48}$$

Note here that c_3 is a $\mathcal{K}_\infty$ function of $c_1 + c_2$, since $\bar{c}$ is a $\mathcal{K}_\infty$ function of c_1. Then, for all $x_{k_0} \in D$, we have:

$$\alpha_1(|x_k|) \le V(k, \mathbf{w}_k, x_0) \le b \ \ \forall\ k \ge k_0 \tag{49}$$
$$\Rightarrow\ \ |x_k| \le \gamma(||\mathbf{w}_k||) + c_3 \ \ \forall\ k \ge k_0 \tag{50}$$

Now consider $x_k \in \mathcal{X}$, $x_k \notin D$. Again, from (2b), Lemma 30, and the definition of b:

$$V(k+1, \mathbf{w}_{k+1}, x_0) - V(k, \mathbf{w}_k, x_0)$$
$$\le -\alpha_3(|x_k|) + \sigma(||\mathbf{w}||) + c_2 \tag{51a}$$
$$\le -\alpha_4(V(k, \mathbf{w}_k, x_0)) + \bar{c} + \sigma(||\mathbf{w}||) + c_2 \tag{51b}$$
$$\le -\hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) + \bar{c} + \sigma(||\mathbf{w}||) + c_2 \tag{51c}$$
$$= -\hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) + \rho \circ \hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) \\ +\bar{c} + \sigma(||\mathbf{w}||) + c_2 - \rho \circ \hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) \tag{51d}$$
$$\le -\hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) + \rho \circ \hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) \tag{51e}$$
$$= -(id - \rho) \circ \hat{\alpha}_4 \circ V(k, \mathbf{w}_k, x_0) \tag{51f}$$
$$\Rightarrow V(k+1, \mathbf{w}_{k+1}, x_0) \\ \le (id - (id - \rho) \circ \hat{\alpha}_4) \circ V(k, \mathbf{w}_k, x_0) \tag{51g}$$
$$\forall\ x_k \notin D,\ x_k \in X,\ \mathbf{w} \in \mathcal{W}$$

Then by Lemma 4.3 of [19], there exists some $\hat{\beta} \in \mathcal{KL}$ such that:

$$V(k, \mathbf{w}_k, x_0) \le \hat{\beta}(V(0, \mathbf{w}_0, x_0), k) \tag{52a}$$
$$\Rightarrow |x_k| \le \alpha_1^{-1}(\hat{\beta}(\alpha_2(|x_0|) + c_1, k)) \tag{52b}$$
$$\forall\ x_k \notin D,\ x_k \in \mathcal{X},\ \mathbf{w} \in \mathcal{W}$$

and by Lemma 31, there exists some $\beta \in \mathcal{KL}$ and $c_4 \in \mathbb{R}_+$ such that:

$$|x_k| \le \beta(|x_0|, k) + c_4 \tag{53}$$
$$\forall\ x_k \notin D,\ x_k \in \mathcal{X},\ \mathbf{w} \in \mathcal{W}$$

Note that, by Lemma 31, c_4 is a $\mathcal{K}_\infty$ function of c_1. Finally, add (53) and (50) together to see that

$$|x_k| \le \beta(|x_0|, k) + \gamma(||\mathbf{w}||) + c_3 + c_4 \tag{54}$$
$$\forall\ \ x_0 \in \mathcal{X},\ \mathbf{w} \in \mathcal{W},\ k \in \mathbb{Z}_+$$

so ISpS is satisfied with $c = c_3 + c_4$ by Definition 7, and c is a $\mathcal{K}_\infty$ function of $c_1 + c_2$. □

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