Study Guide for Model Predictive Control
June 20, 2013

The following material is a compact summary of the essential elements of model predictive control (MPC), and is intended as a study guide to supplement the material in Chapter 27 of Ogunnaike and Ray.

Recall that you may consider two fundamentally different approaches when facing a multivariable control problem:

1. **Divide and Conquer.** Pair inputs and outputs and design simple (PI) controllers for each SISO system.
2. **Model and Optimize.** Model the full multivariable system and design a multivariable controller.

The first approach is simpler and can be used in many process control applications. This approach becomes intractable if the interactions are strong or constraints on the manipulated variables or system states are likely to be important. The second approach requires significant modelling effort, but will pay off if it is used in the economically important parts of the plant that are running at constraints.

1. Elements of MPC

Consider Figure 1, which represents an MPC control problem. I consider the following elements to be the essential elements of MPC.

Dynamic model to make predictions. We use the state-space description of a linear, discrete time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

x is an n-vector, u and y are m-vectors (square plant) and assume the state is measured (the output measurement is addressed in Section 2).

Performance objectives. In current implementations these are stated as a scalar objective function and constraints.

$$V = \sum_{j=0}^{N-1} x(j)'Qx(j) + u(j)'Ru(j)$$

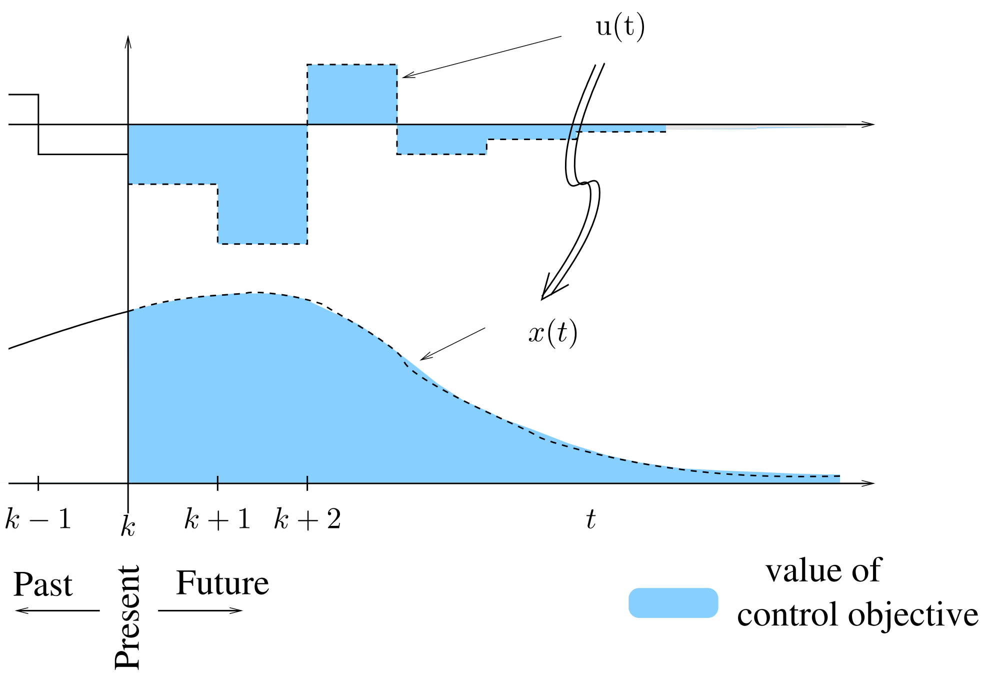

Figure 1: MPC: prediction horizon and objective function.

$$Du(k) \leq d, \qquad k = 0, 1, \ldots N-1$$
$$Hx(k) \leq h, \qquad k = 0, 1, \ldots N-1$$

The matrices Q $(n \times n)$ and R $(m \times m)$ are the primary tuning parameters in this approach, and they are usually chosen to be diagonal matrices. If you wish to have tight control in certain elements of the x-vector, you choose the corresponding diagonal elements of Q large. If you wish to use only small manipulated variable action in some actuator, you choose the corresponding diagonal elements of R large. The loop responds quickly (tight control) when Q's elements are large relative to R's and responds slowly (loose control) with the reverse tuning.

On-line implementation of an algorithm that optimizes the performance objective based on the prediction model. The controller solves the optimization problem

$$\min_{\{u(k)\}} V(u(0), u(1), \ldots u(N)) \tag{1}$$

subject to the model and constraints.

Feedback from the sensors to update the model predictions at the next time step. If we assume we are measuring the state, the solution to the optimization problem (without constraints) is proportional state feedback control

$$u(k) = Kx(k)$$

and the closed-loop system is

$$x(k+1) = (A + BK)x(k)$$

You already examined the effect of the tuning parameters on the eigenvalues of $A+BK$ in homework 10.

2. Output rather than state feedback and offset free control

As we know from our discussions of PID control, proportional feedback control does not necessarily remove steady-state offset, which is normally a desirable objective in process control applications. Also, we do not normally measure the full state of the system (all concentrations, temperatures, pressures, levels, etc.) but only a small subset of states are measured. We can address both of these points in how we use the output measurements in the feedback law.

Integral control is brought into MPC in a rather indirect way. The model for the difference between model predictions and output measurements is the integrating element. Let $d(k)$ be a new variable that holds the difference between our measurement and model prediction at time k,

$$d(k) = y(k) - Cx(k) \tag{2}$$

We now need to predict how this discrepancy will change in the future. The most popular choice in applications it to guess that the discrepancy is constant in the future; in other words,

$$d(k+1) = d(k)$$

As a dynamic system in discrete time, recall that the eigenvalues of this system are all at 1, hence this model is an **integrator**.[1] Although this may not seem like enough, this integrator will ensure offset free control in MPC. With this disturbance model, we augment our prediction model via

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ d(k+1) &= d(k) \\ y(k) &= Cx(k) + d(k) \end{aligned}$$

[1] Put a step input into this model and solve for $d(k)$. Do you see the ramp in $d(k)$?

which can be written as an augmented state-space model,

$$\begin{aligned} \begin{bmatrix} x \\ d \end{bmatrix}(k+1) &= \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}(k) \end{aligned}$$

and we update with the output measurement via Equation 2. Notice that we no longer use state feedback. The output measurement updates the $d(k)$ variable. We modify the predictions using the state space model plus the constant disturbance term. This approach gives us the two desirable features. We use output feedback, and we have an integrator in the augmented system model that ensures no offset in the outputs.

3. Historical remarks

Model predictive control has been an outstanding success in the chemical process industries during the late 1970s through the present, and the industrial practitioners who implemented this technology deserve a great deal of credit for this achievement. However, the early explanations of MPC in the literature tend to obscure what are the essential features of the approach and what are special implementation choices that might change over time.

Another point that was not brought out in the early MPC literature, is the close connection between MPC and the linear quadratic regulator (LQR) of the early 1960s. The essential element that is different is the use of hard constraints in MPC. With the constraints, which are considered important in process control practice, the controller is nonlinear, even though the model is linear. That is a big departure from the LQR, in which the control law is linear and the optimization problem can be solved off-line for an optimal state feedback gain. The MPC optimization problem must be solved on line because there is no closed-form solution. On the other hand, the LQR literature does address infinite horizon problems and does ensure closed-loop stability.

We have now entered a period in which several review articles have pointed out the essential features of MPC, its connections to LQR theory and optimal control theory, and the important role of constraints in process control applications. MPC remains the practitioner's tool of choice for advanced, multivariable and difficult process control problems. As for the future of this approach, it seems that the scale of chemical processes tackled with a single MPC controller is increasing, and we may start to see nonlinear models, based on chemical engineering fundamentals, play a larger role in MPC implementations.