Example Eiganvalues, Poles

$$a_2 \frac{\partial^2 y}{\partial t} + a_1 \frac{\partial y}{\partial t} + a_0 y = 0 \qquad \begin{array}{l} y(0) = \text{given} \\ y'(0) = \text{given} \end{array}$$

$$\frac{\partial^2 y}{\partial t^2} + \frac{a_1}{a_2} \frac{\partial y}{\partial t} + \frac{a_0}{a_2} y = 0$$

=> redefine a_1 & a_0 to avoid notation overload

$$\frac{\partial^2 y}{\partial t^2} + a_1 \frac{\partial y}{\partial t} + a_0 y = 0$$

=> Define $X_1 = y$ => $\frac{\partial X_1}{\partial t} = \frac{\partial y}{\partial t} = X_2$

$X_2 = \frac{\partial y}{\partial t}$

and so

$$\frac{\partial X_2}{\partial t} = -a_1 X_2 - a_0 X_1 \qquad X_2(0) = y'(0)$$

$$\frac{\partial X_1}{\partial t} = X_2 \qquad X_1(0) = y(0)$$

In Matrix Form

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

eiganvalues $\quad \det(A - \lambda I) = 0$

2

$$\left| \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right|$$

$$= \begin{vmatrix} -\lambda & 1 \\ -a_0 & -a_1 - \lambda \end{vmatrix} = \lambda(a_1 + \lambda) + a_0$$

$$= \lambda^2 + a_1\lambda + a_0$$

$$\Rightarrow \boxed{\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}}$$

Now compute eigenvectors

$$A\nu = \lambda\nu \qquad \text{define } \begin{pmatrix} \nu^{(1)} \\ \nu^{(2)} \end{pmatrix} = \nu$$

$$\begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} \nu(1) \\ \nu(2) \end{pmatrix} = \lambda \begin{pmatrix} \nu(1) \\ \nu(2) \end{pmatrix}$$

from first row

$$\nu(2) = \lambda\nu(1)$$

from second row

$$a_0\nu(1) + a_1\nu(2) + \lambda\nu(2) = 0$$

$$\Rightarrow (a_0 + a_1\lambda + \lambda^2)\nu(1) = 0 \Rightarrow \nu(1) \neq 0$$

$$\Rightarrow \boxed{\nu = \begin{pmatrix} \nu(1) \\ \lambda\nu(1) \end{pmatrix}}$$

Solution is thus of the form

$$\boxed{x(t) = \phi \exp(Dt)\, \bar{x}(0)}$$

with $\bar{x}(0) = \phi^{-1} x(0)$

with $\phi = \begin{pmatrix} v_1^{(1)} & v_2^{(1)} \\ \lambda_1 v_1^{(1)} & \lambda_2 v_2^{(1)} \end{pmatrix}$

$D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$

Note that eigenvector is not unique!

but gives same solution

Now get laplace

$$\frac{\partial^2 y}{\partial t^2} + a_1 \frac{\partial y}{\partial t} + a_0 y = 0$$

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = s^2 \hat{y}(s) - s y(0) - y'(0)$$

$$L\left\{\frac{\partial y}{\partial t}\right\} = s\hat{y}(s) - y(0)$$

$$\Rightarrow \underline{s^2 \hat{y}(s)} - s y(0) - y'(0) + \underline{a_1 s \hat{y}(s)} - a_1 y(0) + \underline{a_0 \hat{y}(s)} = 0$$

$$(s^2 + a_1 s + a_0)\hat{y}(s) = (s + a_1) y(0) + y'(0)$$

$$\hat{y}(s) = \frac{(s+a_1) y(0) + y'(0)}{s^2 + a_1 s + a_0}$$

$$= \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2}$$

with $r_1, r_2 = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$

$$A_1(s - r_2) + A_2(s - r_1) = (s + a_1) y(0) + y'(0)$$

$s = r_2$ $\Rightarrow A_2(r_2 - r_1) = (r_2 + a_1) y(0) + y'(0)$

$$A_2 = (r_2 + a_1) y(0) + y'(0) / (r_2 - r_1)$$

$s = r_1 \Rightarrow A_1 (r_1 - r_2) = (r_1 + a_1) y(0) + y'(0)$

$$A_1 = \frac{(r_1 + a_1) y(0) + y'(0)}{r_1 - r_2}$$

$$\Rightarrow \boxed{y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}}$$

See example $\boxed{\text{runsecondorderex1.m}}$