conditions. Even if λ_F and C_p are perfectly known, and both F and T_i can be measured perfectly, T will only equal T^* at steady state. There will be a transient period during which $T \neq T^*$

1.3 The appropriate mathematical model under these conditions is:

$$\frac{dy}{dt} = \frac{\xi}{A_c} - \frac{Ky}{A_c} \qquad (1.2)$$

since $u = Ky$. The appropriate parameter values are:

$\xi = (0.075 - 0.05) = 0.025\ m^3/s$, the disturbance;
$A_c = 1.5\ m^2$; $K = 0.05$.

The solution to (1.2) with these parameters is

$$y(t) = \tfrac{1}{2}\left[1 - \exp(-t/30)\right] \qquad (1.3)$$

As $t \to \infty$ $y(t) \to 0.5m$. Since this represents the <u>change</u> in the liquid level, then we deduce that after steady-state has been achieved, the liquid level in the tank settles to:

$$h = 3 + 0.5 = 3.5\ m$$

Since the liquid level is desired to remain at 3m, there is an offset of $3.5 - 3 = 0.5m$.

1.4 Under these new conditions, the mathematical model becomes:

$$\frac{dy}{dt} = \frac{\xi}{A_c} - \frac{K}{A_c}\left\{y + \int_0^t y(\tau)\,d\tau\right\} \qquad (1.4)$$

Differentiating this equation once, recalling that ξ, A_c, K are constants, we obtain:

$$\frac{d^2y}{dt^2} = -\frac{K}{A_c}\frac{dy}{dt} - \frac{K}{A_c}y(t) \qquad (1\cdot5)$$

At steady state all derivatives vanish and (1·5) reduces to

$$y(t) = 0 \qquad (1\cdot6)$$

implying <u>no change</u> in the liquid level at <u>steady-state</u> and hence <u>no</u> steady-state offset.

1·5 Under these conditions, the original mathematical model becomes

$$A_c\frac{dh}{dt} = F_i - (F_i + \delta)$$

or

$$A_c\frac{dh}{dt} = -\delta$$

whose solution is:

$$\boxed{h(t) = h(0) - \left(\frac{\delta}{A_c}\right)t} \qquad (1\cdot7)$$

Thus $h(t)$, the liquid level, will decrease linearly with time at a rate determined by (δ/A_c). In physical terms the interpretation is as follows:

As a result of <u>overestimating</u> the inlet flow rate by δ (due to flow measurement errors) the feedforward scheme is misled into overcompensation. Material is withdrawn faster than it is actually being introduced into the tank; outflow becomes consistently larger than inflow, and the liquid level therefore <u>drops</u> linearly with time.

CHAPTER 3

$\boxed{3.1}$ (i) By definition

$$\mathcal{L}\{\cos\omega t\} = \int_0^\infty e^{-st}\cos\omega t\,dt \qquad (3.1)$$

Integrating by parts, with

$$u = e^{-st} \Rightarrow du = -se^{-st}$$

$$dv = \cos\omega t \Rightarrow v = \frac{1}{\omega}\sin\omega t$$

we have

$$\int_0^\infty e^{-st}\cos\omega t\,dt = \frac{1}{\omega}e^{-st}(\sin\omega t)\Big|_0^\infty + \int_0^\infty se^{-st}\frac{\sin\omega t}{\omega}\,dt \qquad (3.2)$$

with $u = e^{-st} \Rightarrow du = -se^{-st}$

$$dv = \sin\omega t \Rightarrow v = -\frac{1}{\omega}\cos\omega t$$

we have $\displaystyle\int_0^\infty se^{-st}\frac{\sin\omega t}{\omega} = -\frac{e^{-st}\cos\omega t}{\omega}\Big|_0^\infty - \int_0^\infty \frac{s^2}{\omega^2}e^{-st}\cos\omega t\,dt$

which reduces (3.2) to:

$$\left(1+\frac{s^2}{\omega^2}\right)\int_0^\infty e^{-st}\cos\omega t\,dt = \frac{1}{\omega}e^{-st}(\sin\omega t)\Big|_0^\infty - \left[\frac{1}{\omega}e^{-st}\cos\omega t\right]\Big|_0^\infty$$

$$= \frac{1}{\omega}$$

Thus $$\mathcal{L}\{\cos\omega t\} = \frac{1}{\omega}\cdot\frac{1}{1+s^2/\omega^2} \qquad (3.3)$$

or $$\boxed{\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}}$$ as required.

$\boxed{2.7}$ For converter noted

$$I = 204.7\,(V)$$

(a) $I = -204.8 \cong \underline{-205}$

(b) $I = 204.7(2.7) = 552.6 \cong \underline{553}$

(c) $I = 204.7(8.7) = 1780.89 \cong \underline{1781}$

(d) $I = -204.8(6) = -1228.2 \cong -\underline{1228}$.

$\boxed{2.8}$ (a) 2400 baud $\equiv$ 240 cps

FORTRAN program = 500 lines x 30 char/line = 15000 char.

Time = 15000 char/240 cps = 62.5 seconds.

(b) For 9600 baud line $\equiv$ 960 cps

Time = 15000 char/960 cps

= 15.63 seconds.

(ii) $\mathcal{L}\{\sin \omega t\} = \int_0^\infty e^{-st} \sin \omega t \, dt$

Integrating by parts gives:

$$\int_0^\infty e^{-st} \sin \omega t \, dt = -\frac{1}{\omega} e^{-st} \cos \omega t \Big|_0^\infty - \frac{s}{\omega} \int_0^\infty e^{-st} \cos \omega t \, dt$$

$$= -\frac{1}{\omega}(0 - 1) - \frac{s}{\omega}\underbrace{\left(\frac{s}{s^2+\omega^2}\right)}_{\text{Result from part (i)}}$$

Simplifying further gives

$$\boxed{\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}}$$ as required.

3·2 To establish that

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} \quad (3{\cdot}4)$$

Observe that, by definition,

$$\text{LHS} = \int_0^\infty [c_1 f_1(t) + c_2 f_2(t)] e^{-st} \, dt$$

$$= \int_0^\infty c_1 f_1(t) e^{-st} dt + \int_0^\infty c_2 f_2(t) e^{-st} dt$$

$$= c_1 \int_0^\infty f_1(t) e^{-st} dt + c_2 \int_0^\infty f_2(t) e^{-st} dt$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} = \text{RHS}, \text{ as reqd.}$$

3·3 (i) $\mathcal{L}\{\cos \omega t\} = \frac{1}{2}\mathcal{L}\{(e^{j\omega t} + e^{-j\omega t})\}$

$$= \frac{1}{2}\left(\frac{1}{s-j\omega}\right) + \frac{1}{2}\left(\frac{1}{s+j\omega}\right) \quad (3{\cdot}5)$$

since $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

Simplifying (3.5) further gives:

$$\mathcal{L}\{\cos \omega t\} = \frac{1}{2}\,\frac{2s}{s^2+\omega^2}$$

$$= \frac{s}{s^2+\omega^2} \quad , \text{ as required.}$$

ii) Similarly

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{2j}\,\mathcal{L}\{e^{j\omega t} - e^{-j\omega t}\}$$

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega}\right) - \frac{1}{2j}\left(\frac{1}{s+j\omega}\right)$$

$$= \frac{1}{2j}\quad \frac{2j\omega}{s^2+\omega^2}$$

$$= \frac{\omega}{s^2+\omega^2} \quad , \text{ as required.}$$

$\boxed{3.4}$ Given $\quad f(t) = t^n \qquad (4.1)$

to find $\quad \bar{f}(s) = \mathcal{L}\{f(t)\}$

proceed by differentiating $f(t)$ n times:

$$f(t) = t^n \quad \Rightarrow \quad f(0) = 0$$

and

$$f'(t) = \frac{df}{dt} = n t^{n-1} \; ; \; \Rightarrow \; f'(0) = 0$$

$$f''(t) = \frac{d^2f}{dt^2} = n(n-1)t^{n-2} \; ; \; \Rightarrow \; f''(0) = 0$$

$$\vdots$$

$$f^{(k)}(t) = \frac{d^kf}{dt^k} = n(n-1)\cdots(n-k+1)t^{n-k} \; ; \; \Rightarrow f^{(k)}(0) = 0$$

$$\vdots$$

finally

$$f^{(n)}(t) = \frac{d^nf}{dt^n} = n(n-1)(n-2)\cdots 2\cdot 1$$

$$= \Gamma(n+1) \quad \text{a constant} \qquad (4.2)$$

In this case,

$$E_{accumulated} = mC_{p_2}\frac{dT_2}{dt}$$

$$E_{in} = Q$$

$$E_{out} = \underbrace{A_c h(T_2 - T_a)}_{\text{loss to atmosphere}} + \underbrace{c(T_2 - T_1)}_{\text{Transferred to water in kettle.}}$$

Thus, the energy balance becomes

$$\boxed{mC_{p_2}\frac{dT_2}{dt} = Q - A_c h(T_2 - T_a) - c(T_2 - T_1)} \quad (4.6)$$

$\boxed{4.3}$ (a) Input variable : T_2
Disturbance variable : T_0

(b) At steady state (P4.1) becomes

$$0 = c(T_2^* - T_1^*) + F\rho C_p T_0^* - F\rho C_p T_1^*$$

Subtracting from (P4.1), with $y = T - T_1^*$; $u = T_2 - T_2^*$, and $d = T_0 - T_1^*$, gives

$$\rho V C_p \frac{dy}{dt} = c(u - y) + F\rho C_p d - F\rho C_p y$$

Rearrange to obtain:

$$\frac{dy}{dt} = -\left(\frac{F}{V} + \frac{c}{\rho V C_p}\right)y + \frac{c}{\rho V C_p}u + \frac{F}{V}d \quad (4.7)$$

which is of the form given in (P4.2) with

$$a = -\left(\frac{F}{V} + \frac{c}{\rho V C_p}\right); \quad b = \frac{c}{\rho V C_p}; \quad \gamma = \frac{F}{V}$$

(c) Take Laplace transforms in (4.7) and rearrange to give:

$$y(s) = \underbrace{\left[\frac{c/\rho V C_p}{s + \left(\frac{F}{V} + \frac{c}{\rho V C_p}\right)}\right]}_{g(s)} u(s) + \underbrace{\left[\frac{F/V}{s + \left(\frac{F}{V} + \frac{c}{\rho V C_p}\right)}\right]}_{g_d(s)} d(s)$$

The required transfer functions are as indicated.

(d) For the current purposes, it is convenient to let:

$$k = \frac{\frac{c}{\rho V C_p}}{\frac{F}{V} + \frac{c}{\rho V C_p}} \; ; \quad \text{i.e.} \quad k = \frac{c}{F\rho C_p + c} \qquad (4.8a)$$

$$k_d = \frac{F/V}{\frac{F}{V} + \frac{c}{\rho V C_p}} \; ; \quad \text{i.e.} \quad k_d = \frac{F\rho C_p}{F\rho C_p + c} \qquad (4.8b)$$

$$\tau = \frac{1}{\left(\frac{F}{V} + \frac{c}{\rho V C_p}\right)} \; ; \quad \text{i.e.} \quad \tau = \frac{\rho V C_p}{F\rho C_p + c} \qquad (4.8c)$$

Then

$$g(s) = k/\tau s + 1 \quad \Rightarrow \quad g(t) = \frac{k}{\tau} e^{-t/\tau} \qquad (4.9a)$$

$$g_d(s) = \frac{k_d}{\tau s + 1} \quad \Rightarrow \quad g_d(t) = \frac{k_d}{\tau} e^{-t/\tau} \qquad (4.9b)$$

The impulse response model is therefore:

$$\boxed{y(t) = \int_0^t \frac{c}{\rho V C_p} \exp\left\{-\frac{(t-\sigma)}{\tau}\right\} u(\sigma)\, d\sigma + \int_0^t \frac{F}{V} \exp\left\{-\frac{(t-\sigma)}{\tau}\right\} d(\sigma)\, d\sigma}$$

(4.10)