Problem 1: Derivation of equations (20.18) and (20.19) from Ogunnaike and Ray

We have the following dynamic system:

$$A_c \frac{dh}{dt} = F_H + F_C + F_d - K\sqrt{h} \tag{1}$$

$$A_c \frac{d(hT)}{dt} = F_H T_H + F_C T_C + F_d T_d - K\sqrt{h}\,T \tag{2}$$

We wish to linearize the system about the following steady-state point:

$$\begin{aligned} h &= h_s & T &= T_s \\ F_H &= F_{Hs} & F_C &= F_{Cs} \\ F_d &= F_{ds} & T_d &= T_{ds} \end{aligned}$$

As such, we define the following deviation variables:

$$\begin{aligned} x_1 &= h - h_s & x_2 &= T - T_s \\ u_1 &= F_H - F_{Hs} & u_2 &= F_C - F_{Cs} \\ d_1 &= F_d - F_{ds} & d_2 &= T_d - T_{ds} \end{aligned}$$

We begin by rearranging equation 2:

$$\begin{aligned} A_c h \frac{dT}{dt} + A_c T \frac{dh}{dt} &= F_H T_H + F_C T_C + F_d T_d - K\sqrt{h}\,T \\ A_c h \frac{dT}{dt} &= \left[F_H T_H + F_C T_C + F_d T_d - K\sqrt{h}\,T - A_c T \frac{dh}{dt} \right] \\ A_c \frac{dT}{dt} &= \frac{1}{h}\left[F_H T_H + F_C T_C + F_d T_d - K\sqrt{h}\,T - A_c T \frac{dh}{dt} \right] \\ &= \frac{1}{h}\left[F_H T_H + F_C T_C + F_d T_d - K\sqrt{h}\,T - T(F_H + F_C + F_d - K\sqrt{h}) \right] \\ &= \frac{1}{h}\left[F_H (T_H - T) + F_C (T_C - T) + F_d (T_d - T) \right] \end{aligned} \tag{3}$$

Next we define f_1 and f_2 as follows:

$$\begin{aligned} f_1(h, F_H, F_C, F_d) &= F_H + F_C + F_d - K\sqrt{h} \\ f_2(h, T, F_H, F_C, F_d, T_d) &= \frac{1}{h}\left[F_H (T_H - T) + F_C (T_C - T) + F_d (T_d - T) \right] \end{aligned}$$

We will proceed by first linearizing f_1 and f_2 and then rewriting the result in terms of deviation variables. Let $f_{1,l}$ and $f_{2,l}$ denote the linearized forms of f_1 and f_2 respectively.

$\underline{f_{1,l}}$

As explained in section 10.4 of Ogunnaike and Ray, we linearize $f_{1,l}$ with a first order Taylor series expansion:

$$\begin{aligned}
f_{1,l}(h, F_H, F_C, F_d) &= f_1(h_s, F_{Hs}, F_{Cs}, F_{ds}) \\
&\quad + \left(\frac{\partial f_1}{\partial h}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(h - h_s) + \left(\frac{\partial f_1}{\partial F_H}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(F_H - F_{Hs}) \\
&\quad + \left(\frac{\partial f_1}{\partial F_C}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(F_C - F_{Cs}) + \left(\frac{\partial f_1}{\partial F_d}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(F_d - F_{ds}) \\
&= F_{Hs} + F_{Cs} + F_{ds} - K\sqrt{h_s} \\
&\quad - \frac{K}{2\sqrt{h_s}}(h - h_s) + (F_H - F_{Hs}) + (F_C - F_{Cs}) + (F_d - F_{ds}) \\
&= -K\sqrt{h_s} - \frac{K}{2\sqrt{h_s}}(h - h_s) + F_H + F_C + F_d
\end{aligned}$$

The steady-state version of equation 1 is:

$$0 = -K\sqrt{h_s} + F_{Hs} + F_{Cs} + F_{ds}$$

Subtracting this from the equation above, we get:

$$f_{1,l}(h, F_H, F_C, F_d) = -\frac{K}{2\sqrt{h_s}}(h - h_s) + (F_H - F_{Hs}) + (F_C - F_{Cs}) + (F_d - F_{ds})$$

Therefore we have:

$$A_c\frac{dh}{dt} \approx -\frac{K}{2\sqrt{h_s}}(h - h_s) + (F_H - F_{Hs}) + (F_C - F_{Cs}) + (F_d - F_{ds})$$

Rewriting in terms of deviation variables gives the final result:

$$A_c\frac{dx_1}{dt} \approx -\frac{K}{2\sqrt{h_s}}x_1 + u_1 + u_2 + d_1$$

$$\boxed{\frac{dx_1}{dt} \approx \frac{1}{A_c}\left[-\frac{K}{2\sqrt{h_s}}x_1 + u_1 + u_2 + d_1\right]}$$

Note that we have used the following fact:

$$\frac{dh}{dt} = \frac{dx_1}{dt}$$

$\underline{f_{2,l}}$

As explained in section 10.4 of Ogunnaike and Ray, we linearize $f_{2,l}$ with a first order Taylor series expansion:

$$\begin{aligned}
f_{2,l}(h,T,F_H,F_C,F_d,T_d) &= f_2(h_s,T_s,F_{Hs},F_{Cs},F_{ds},T_{ds}) \\
&\quad + \left(\frac{\partial f_2}{\partial h}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(h-h_s) + \left(\frac{\partial f_2}{\partial T}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(T-T_s) \\
&\quad + \left(\frac{\partial f_2}{\partial F_H}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(F_H-F_{Hs}) + \left(\frac{\partial f_2}{\partial F_C}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(F_C-F_{Cs}) \\
&\quad + \left(\frac{\partial f_2}{\partial F_d}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(F_d-F_{ds}) + \left(\frac{\partial f_2}{\partial T_d}\right)_{(h_s,F_{Hs},F_{Cs},F_{ds})}(T_d-T_{ds}) \\
&= \frac{1}{h_s}\left[F_{Hs}(T_H-T_s)+F_{Cs}(T_C-T_s)+F_{ds}(T_{ds}-T_s)\right] \\
&\quad - \frac{1}{h_s^2}\left[F_{Hs}(T_H-T_s)+F_{Cs}(T_C-T_s)+F_{ds}(T_{ds}-T_s)\right](h-h_s) \\
&\quad + \frac{1}{h_s}(-F_{Hs}-F_{Cs}-F_{ds})(T-T_s) + \frac{1}{h_s}(T_H-T_s)(F_H-F_{Hs}) \\
&\quad + \frac{1}{h_s}(T_C-T_s)(F_C-F_{Cs}) + \frac{1}{h_s}(T_{ds}-T_s)(F_d-F_{ds}) + \frac{1}{h_s}(F_{ds})(T_d-T_{ds})
\end{aligned}$$

The steady state version of equation 3 is:

$$\begin{aligned}
0 &= \frac{1}{h_s}\left[F_{Hs}(T_H-T_s)+F_{Cs}(T_C-T_s)+F_{ds}(T_{ds}-T_s)\right] \\
0 &= F_{Hs}(T_H-T_s)+F_{Cs}(T_C-T_s)+F_{ds}(T_{ds}-T_s)
\end{aligned}$$

The steady-state version of equation 1 is:

$$\begin{aligned}
0 &= -K\sqrt{h_s}+F_{Hs}+F_{Cs}+F_{ds} \\
-F_{Hs}-F_{Cs}-F_{ds} &= -K\sqrt{h_s}
\end{aligned}$$

Substituting these two results into the above equation gives:

$$\begin{aligned}
f_{2,l}(h,T,F_H,F_C,F_d,T_d) &= -\frac{1}{h_s}K\sqrt{h_s}(T-T_s) + \frac{1}{h_s}(T_H-T_s)(F_H-F_{Hs}) \\
&\quad + \frac{1}{h_s}(T_C-T_s)(F_C-F_{Cs}) + \frac{1}{h_s}(T_{ds}-T_s)(F_d-F_{ds}) + \frac{1}{h_s}(F_{ds})(T_d-T_{ds})
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
A_c\frac{dT}{dt} &\approx -\frac{1}{h_s}K\sqrt{h_s}(T-T_s) + \frac{1}{h_s}(T_H-T_s)(F_H-F_{Hs}) + \frac{1}{h_s}(T_C-T_s)(F_C-F_{Cs}) \\
&\quad + \frac{1}{h_s}(T_{ds}-T_s)(F_d-F_{ds}) + \frac{1}{h_s}(F_{ds})(T_d-T_{ds}) \\
&= \frac{1}{h_s}\Big[-K\sqrt{h_s}(T-T_s)+(T_H-T_s)(F_H-F_{Hs})+(T_C-T_s)(F_C-F_{Cs}) \\
&\quad + (T_{ds}-T_s)(F_d-F_{ds})+F_{ds}(T_d-T_{ds})\Big]
\end{aligned}$$

Rewriting in terms of deviation variables gives the final result:

$$A_c \frac{dx_2}{dt} \approx \frac{1}{h_s}\left[-K\sqrt{h_s}\,x_2 + (T_H - T_s)u_1 + (T_C - T_s)u_2 + (T_{ds} - T_s)d_1 + F_{ds}d_2\right]$$

$$\boxed{\frac{dx_2}{dt} \approx \frac{1}{A_c h_s}\left[-K\sqrt{h_s}\,x_2 + (T_H - T_s)u_1 + (T_C - T_s)u_2 + (T_{ds} - T_s)d_1 + F_{ds}d_2\right]}$$

Note that we have used the following fact:

$$\frac{dT}{dt} = \frac{dx_2}{dt}$$

(b) From the impulse response model, the required response is obtained from

$$y(t) = \int_0^t \underline{G}(t-\sigma)\begin{bmatrix}1\\0\end{bmatrix} d\sigma$$

or:

$$y_1(t) = \int_0^t \frac{2}{3}\left(2e^{-2(t-\sigma)} + e^{-5(t-\sigma)}\right] d\sigma \quad (20.4)$$

$$y_2(t) = \int_0^t \frac{2}{3}\left[e^{-2(t-\sigma)} - e^{-5(t-\sigma)}\right] d\sigma \quad (20.5)$$

Perform the required integration in Eq (20.4) above to obtain

$$y_1(t) = \frac{2}{3}\left(1-e^{-2t}\right) + \frac{2}{15}\left(1-e^{-5t}\right) \quad (20.6)$$

and similarly, from Eq (20.5) above, obtain

$$y_2(t) = \frac{1}{3}\left(1-e^{-2t}\right) - \frac{2}{15}\left(1-e^{-5t}\right) \quad (20.7)$$

These same equations are obtainable directly from the transfer function matrix given in Eq (20.1) in part (b) of Problem 20.1.

$\boxed{20.3}$ (a). The eigenvalues of the system matrix $\underset{\sim}{A}$ are obtained from

$$\begin{vmatrix}(4-\lambda) & -1\\ 5 & (-2-\lambda)\end{vmatrix} = 0 \;;\; \text{or} \quad \lambda^2 - 2\lambda - 3 = 0$$

as $\lambda_1 = +3$; $\lambda_2 = -1$
and the positive eigenvalue confirms that the system is unstable at this operating condition.

(b) The required transfer function is obtained from

$$\underset{\sim}{G}(s) = \begin{bmatrix} (s-4) & 1 \\ -5 & (s+2) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

or

$$\underset{\sim}{G}(s) = \frac{1}{(s-3)(s+1)} \begin{bmatrix} (s+2) & -2 \\ 5 & 2(s-4) \end{bmatrix}.$$

The poles are located at $s=3$; $s=-1$ (same as the eigenvalues of $\underset{\sim}{A}$, of course); there are no zeros. cancels out with denominator

(c) With the given controller expression $\underset{\sim}{u} = -\underset{\sim}{K}_c \underset{\sim}{x}$, the model equation becomes

$$\dot{\underset{\sim}{x}} = (\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{K}_c)\underset{\sim}{x}$$

in the closed loop; or,

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}_{CL}\underset{\sim}{x} \qquad (20.8a)$$

where

$$\underset{\sim}{A}_{CL} = (\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{K}_c) \qquad (20.8b)$$

Given $K_{c_2} = 2$,

$$\underset{\sim}{A}_{CL} = \begin{bmatrix} (4-K_{c_1}) & -1 \\ 5 & -6 \end{bmatrix} \qquad (20.9)$$

The eigenvalues of $\underset{\sim}{A}_{CL}$ are obtained from

$$\begin{vmatrix} (4-K_{c_1})-\lambda & -1 \\ 5 & -6-\lambda \end{vmatrix} = 0$$

or

$$\lambda^2 + (K_{c_1}+2)\lambda + (6K_{c_1} - 19) = 0 \qquad (20.10)$$

For the closed-loop system to be stable, all eigenvalues of $\underset{\sim}{A}_{CL}$ must lies in the LHP, requiring that all the coefficients of the quadratic equation in (20.10) above be positive. Thus for closed-loop stability

$$\boxed{K_{c_1} > 19/6.}$$

$\boxed{20.4}$ (a) The poles of the transfer function matrix are located at

$$s = -3; -4; -1; -7$$

the collection of the poles of the individual elements of the matrix.

To obtain the zeros, find the roots of

$$|G(s)| = 0$$

i.e

$$\frac{1}{(s+3)(s+4)} - \frac{1}{(s+1)(s+7)} = 0$$

or

$$\frac{(s-5)}{(s+3)(s+4)(s+1)(s+7)} = 0 \qquad (20.11)$$

And the zero is located at $s = +5$, i.e. it is

Eq (20.20) above may be rewritten as:

$$|\underline{W}||\underline{W}^T||\underline{I} + \underline{\Sigma}\underline{G}_c^{\Sigma}| = 0 \qquad (20\text{-}21)$$

and since $|\underline{W}| = \dfrac{1}{|\underline{W}^T|}$, (20.21) above reduces immediately to

$$|\underline{I} + \underline{\Sigma}\underline{G}_c^{\Sigma}| = 0 \quad \text{as required.}$$

$\boxed{20.10}$ (a) The characteristic equation to be used in investigating stability is

$$|\underset{\sim}{I} + \underline{G}\underline{G}_c| = 0$$

In this case, with $\underline{G}$ as given in eqn (P20.14),

$$(\underset{\sim}{I} + \underline{G}\underline{G}_c) = \begin{bmatrix} \dfrac{2K_{c_1}}{(s+1)} + 1 & \dfrac{K_{c_2}}{(3s+1)} \\ \dfrac{4K_{c_1}}{(s+1)} & 1 + \dfrac{K_{c_2}}{(2s+1)} \end{bmatrix}$$

and for $K_{c_1} = 1$, $K_{c_2} = 5$, obtain

$$(\underset{\sim}{I} + \underline{G}\underline{G}_c) = \begin{bmatrix} \dfrac{s+3}{s+1} & \dfrac{5}{3s+1} \\ \dfrac{4}{s+1} & \dfrac{2s+6}{2s+1} \end{bmatrix}$$

Upon taking the determinant and rearranging, obtain the characteristic equation:

$$3s^3 + 19s^2 + 13s - 1 = 0 \qquad (20.22)$$

and the presence of the -1 term is enough to indicate that the system with Eq (20.22) above as its characteristic equation is UNSTABLE.

(b) When the pairing is switched, the process model becomes

$$\bar{G}(s) = \begin{bmatrix} \frac{1}{3s+1} & \frac{2}{s+1} \\ \frac{1}{2s+1} & \frac{4}{s+1} \end{bmatrix}$$

and in this case, for the same $K_{c_1} = 1$, $K_{c_2} = 5$,

$$\left(\underset{\sim}{I} + \bar{\underset{\sim}{G}}\underset{\sim}{G}_c\right) = \begin{bmatrix} \frac{3s+2}{3s+1} & \frac{10}{s+1} \\ \frac{1}{2s+1} & \frac{s+21}{s+1} \end{bmatrix}$$

with the characteristic equation:

$$6s^3 + 133s^2 + 119s + 32 = 0 \qquad (20.23)$$

For this system to be stable, Routh's criterion indicates that the following condition must hold:

$$133 \times 119 > 6 \times 32$$

Since this inequality is true, we conclude that the system is indeed stable.

It is interesting to note that the RG parameter for the original $y_1 - m_1 / y_2 - m_2$ pairing is -1.

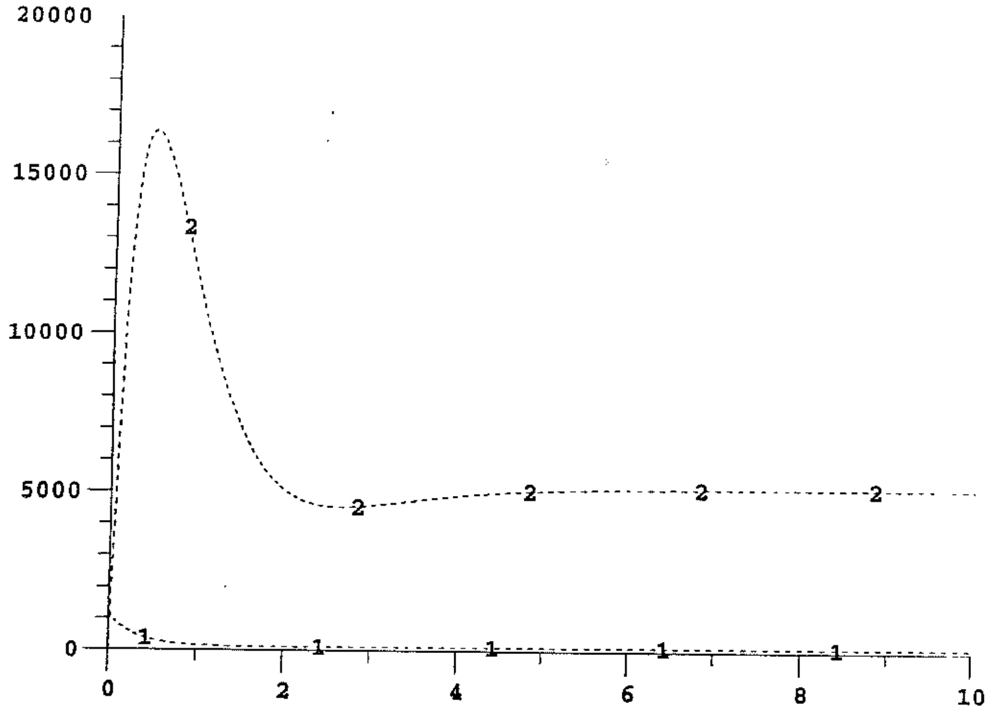

FIG S21.1(b) Manipulated Variables.

$\boxed{21.3}$ The steady state gain matrix for the process is

$$\underset{\sim}{K} = \begin{bmatrix} 0.7 & -0.005 \\ -34.7 & 0.9 \end{bmatrix} \qquad (21.3)$$

from which the RGA is immediately obtained as:

$$\Lambda = \begin{array}{c c c} & m_1 & m_2 & \\ y_1 & \left[\begin{array}{c} 1.38 \\ -0.38 \end{array}\right. & \left.\begin{array}{c} -0.38 \\ 1.38 \end{array}\right] & \begin{array}{c} m_1 \\ m_2 \end{array} \\ y_2 & & & \end{array}$$

indicating a $y_1 - m_1$, $y_2 - m_2$ pairing, and a resulting control system that will experience some interactions.