Optimization Example Problems

CBE470: Process Dynamics and Control - Spring 2017

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`http://zavalab.engr.wisc.edu/teaching/cbe470spring2017`

Problem 1: An interesting observation.

Prove that the optimal solution $x^* \in \Re$ of the optimization problem

$$\min_x \quad f(x) \qquad \text{where} \quad f(x) = \frac{1}{2}\sum_{k=1}^{N}(x-d_k)^2$$

is the average of the data points $d_1, d_2, ..., d_N$. How is the optimal objective value $f(x^*)$ related to the variance of the data points?

Problem 2: Quadratic programs are fun

Derive optimality conditions and find, by hand, the solution (x_1^*, x_2^*) and the optimal objective function of the optimization problem:

$$\begin{aligned} \min\ & (x_1-1)^2 + (x_1-x_2)^2 + (x_2-1)^2 \\ \text{s.t. } & x_1 + x_2 = 2. \end{aligned}$$

Derive optimality conditions and find, by hand, the solution and optimal objective value if, in addition, you impose the constraint $x_1 \geq 2$:

$$\begin{aligned} \min\ & (x_1-1)^2 + (x_1-x_2)^2 + (x_2-1)^2 \\ \text{s.t. } & x_1 + x_2 = 2 \\ & x_1 \geq 2. \end{aligned}$$

Could you have guessed the solutions without using the optimality conditions? How?

Problem 3: Model Predictive Control (MPC)

Assume that you have a differential equation:

$$\dot{x} = ax + bu$$

defined over the time interval $[0, T]$ and with initial condition $x(0) = 0$. Assume now that you discretize this differential equation using N time steps to obtain the recursion:

$$\frac{x_{k+1} - x_k}{h} = ax_k + bu_k,\ k = 0, ..., N-1.$$

This can be rewritten as,

$$x_{k+1} = \bar{a}x_k + \bar{b}u_k,\ k = 0, ..., N-1$$

where $\bar{a} = (1 + ha)$, $\bar{b} = hb$, $h = T/N$, and $x_0 = x(0)$.

- Consider now the MPC problem:

$$\begin{aligned} \min \quad & \sum_{k=0}^{N}(x_k-1)^2 + \sum_{k=0}^{N-1} u_k \\ \text{s.t.} \quad & x_{k+1} = \bar{a}x_k + \bar{b}u_k,\ k = 0, ..., N-1 \end{aligned}$$

  Derive the optimality conditions and derive, by hand, the optimal control (u_0, u_1) and state trajectories (x_1, x_2). Note that x_0 is not a variable but a fixed parameter and $N = 2$.

- Repeat the procedure for $N = 3$. Do you see a trend? Can you generalize this to solve the MPC problem for any N? How?

Problem 4: Multivariable MPC.

Use same procedure of Problem 3 to discretize the multivariable system

$$\dot{x} = Ax + Bu + \Gamma d$$

to obtain the recursion

$$x_{k+1} = \bar{A}x_k + \bar{B}u_k + \bar{\Gamma}d_k,\ k = 0, ..., N-1$$

What are the matrices $\bar{A}$, $\bar{B}$, and $\bar{\Gamma}$?

Assume that $N = 2$, that the dynamic system has 2 states, 2 controls, and 1 disturbance, and that the initial conditions are $x_0^T = x(0)^T = [0, 0]$. Under these assumptions, express the MPC problem:

$$\begin{aligned} \min \quad & \sum_{k=0}^{N}(x_k - 1)^T(x_k - 1) + \sum_{k=0}^{N-1} e^T u_k \\ \text{s.t.} \quad & x_{k+1} = \bar{A}x_k + \bar{B}u_k + \bar{\Gamma}d_k,\ k = 0, ..., N-1 \end{aligned}$$

as a general quadratic program (QP) of the form:

$$\begin{aligned} \min \quad & \frac{1}{2}w^T Q w + c^T w \\ \text{s.t.} \quad & Dw = r. \end{aligned}$$

What are the variable vector w, the data vectors c, r, and the data matrices Q and D? What are their dimensions?

Note: $e^T = [1, 1]$.