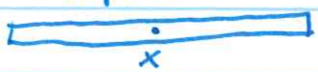
Transient Diffusion via Fourier (Example 3.12)

$$\boxed{U_t = D\,U_{xx}}$$

Temperature



$x \in (-\infty, \infty)$

$U(x,0) = U_0(x) \Rightarrow IC$

$U(-\infty,t) = U(\infty,t) = 0 \Rightarrow BC's$

Recall, Fourier Transform:

$$\bar{F}(j\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x}\,dx \quad \text{or} \quad \overset{\text{Laplace}}{\bar{f}(s)} = \int_0^{\infty} f(x) e^{-sx}\,dx$$

$$= \mathcal{F}\{f(x)\} = \mathcal{L}\{f(x)\}\big|_{s=j\omega} \qquad = \mathcal{L}(f(x))$$

and inverse Fourier transform:

$$f(x) = \mathcal{F}^{-1}\{\bar{F}(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(j\omega) e^{j\omega x}\,d\omega$$

Recall also that:

$$\int_{-\infty}^{\infty} f(x')\,g(x-x')\,dx' = \mathcal{F}^{-1}\{\bar{F}(j\omega)\,\bar{g}(j\omega)\}$$

→ convolution property

and that

$$\int_{-\infty}^{\infty} f(x')\,\delta(x-x')\,dx' = f(x) \Rightarrow \text{delta function}$$

$\Rightarrow$ Applying FT to PDE we get: (or Laplace & then evaluate with $s=j\omega$) ← in space domain

$$\mathcal{F}\{U_t - D U_{xx}\} = \mathcal{F}\{U_t(x,t) - D U_{xx}(x,t)\} = 0$$

$$= \bar{U}_t(j\omega,t) - D(j\omega)^2\,\bar{U}(j\omega,t) = 0$$

where recall

$$\mathcal{F}\left\{\frac{d^n f(x)}{dx^n}\right\} = (j\omega)^n \bar{F}(j\omega)$$

we thus have: $(j\omega)^2 = \sqrt{-1}^2 w^2 = -w^2$

$$\bar{U}_t(j\omega,t) + D\omega^2 \bar{u}(j\omega,t) = 0$$

this is a 1st order ODE with solution:

$$\boxed{\bar{u}(j\omega,t) = \bar{u}(j\omega,0)\, e^{-D\omega^2 t}}$$

where $\bar{u}(j\omega,0) = \mathcal{F}\{U_0(x)\}$ $\Rightarrow$ solution of PDE is: $\boxed{u(x,t) = \mathcal{F}^{-1}\{\bar{U}(j\omega,0)\, e^{D(j\omega)^2 t}\}}$

$\Rightarrow$ Consider $\underbrace{U_0(x) = \delta(x)}_{\mathcal{F}(\delta(x)) = 1} \Rightarrow \bar{u}(j\omega,0) = 1$

Applying convolution:

$$u(x,t) = \int_{-\infty}^{\infty} u(x',t)\,\delta(x-x')\,dx = \mathcal{F}^{-1}\{1 \cdot e^{D(j\omega)^2 t}\} \rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-D\omega^2 t}\right)\left(e^{j\omega x}\right) d\omega$$

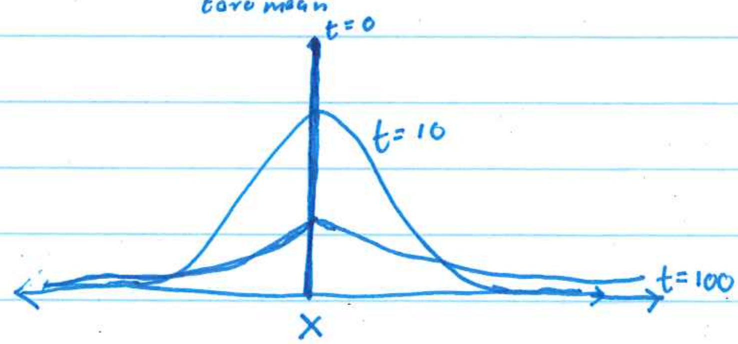
can be obtained by integration by parts with $\int_{-\infty}^{\infty} u(x,t) = 1$ $\Downarrow$ property of Gaussian distribution

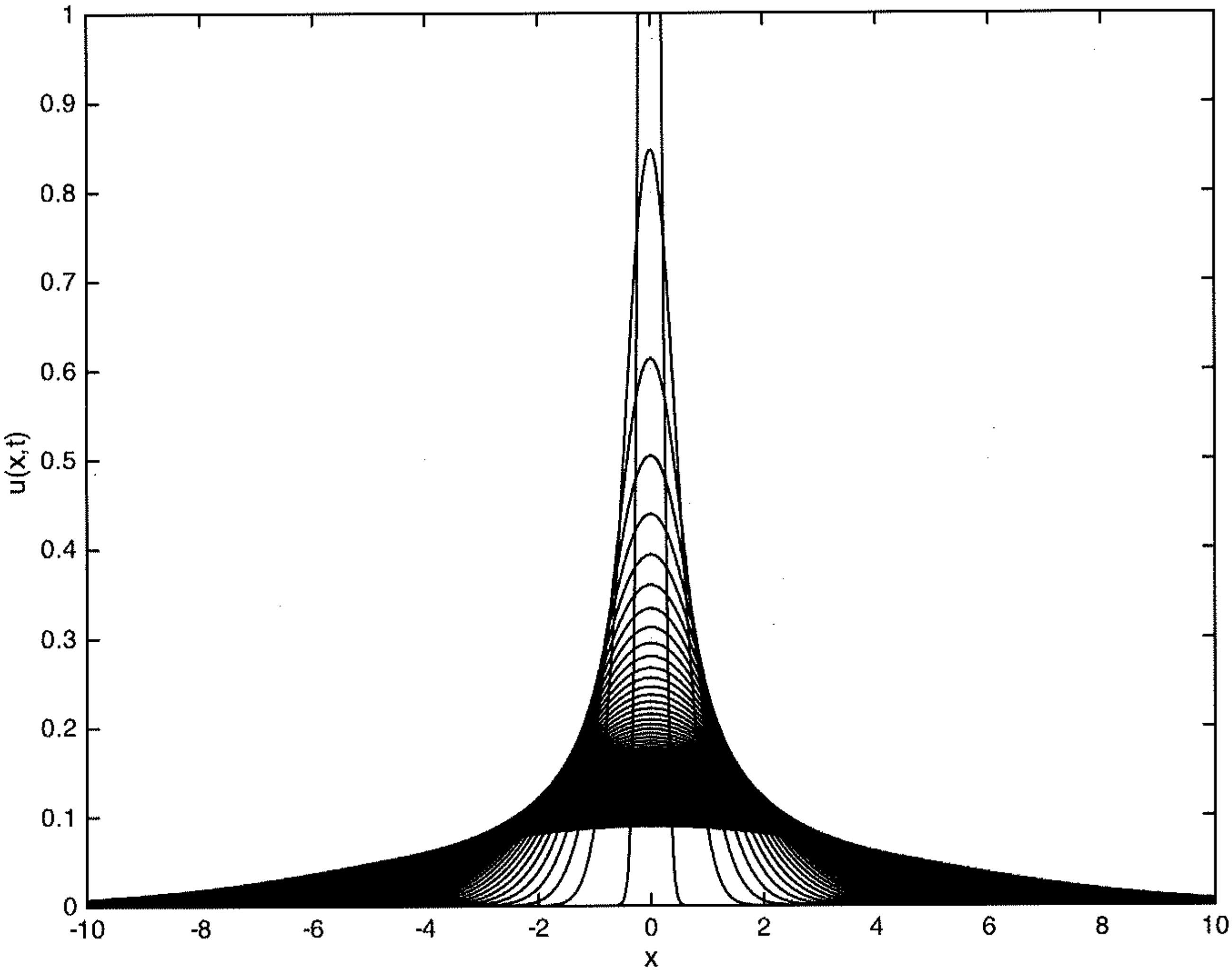
$$\Rightarrow \quad \boxed{u(x,t) = \frac{1}{2\sqrt{\pi D t}}\, e^{-x^2/4Dt}}$$

$\Rightarrow$ starting from $U_0(x) = \delta(x)$, the temperature field is a gaussian distribution with height ~ $\frac{1}{2\sqrt{\pi D t}}$ and width ~ $\sqrt{4Dt}$

$\Downarrow$ Zero mean

$\Downarrow$ standard deviation




u(x,t)
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-8
-6
-4
-2
0
2
4
6
8
10