APPENDIX

C

LAPLACE AND z-TRANSFORMS

Integral transforms have had a most enduring impact on the analysis and design of linear control systems and are key components of the engineer's toolbox. The most popular of these have been the Laplace transform for continuous systems, and the z-transform for discrete systems. Even though certain aspects of the Laplace and z-transforms have been introduced in Chapters 3 and 24, the material presented here is intended to be a more complete summary. A brief introduction, first to the general family of integral transforms, and then to the Fourier transform in particular, precedes the main subject matter; it is designed to emphasize the close ties between the Laplace transform and the Fourier transform. The unifying theme is extended finally to the z-transform by presenting it as a natural consequence of applying the Laplace transform to sampled functions.

C.1 INTRODUCTION: THE GENERAL INTEGRAL TRANSFORM

The process of multiplying a function of a single variable t, say, $f(t)$, by $K(s,t)$, a function of two variables s and t, and integrating the resulting product *with respect to t* between limits a and b, gives rise to a function of the single surviving variable s, say, $\hat{f}(s)$; the original function, $f(t)$, is thus "transformed" to $\hat{f}(s)$ by the indicated process that may be represented mathematically as:

$$\mathrm{T}\{f(t)\} = \int_a^b K(s,t)\, f(t)\, dt = \hat{f}(s) \tag{C.1}$$

This is an expression for the *general* integral transform; $K(s, t)$, the function of two variables that serves as the multiplier in the transform formula is called the *kernel* of the transform.

By specifying various kernels and limits of integration, several useful integral transforms can be obtained from Eq. (C.1). For example, when $a = 0$, $b = \infty$, and $K(s, t) = e^{-st}$, we have the Laplace transform defined in Eq. (3.1). Some other examples are:

1. FOURIER TRANSFORM: $a = -\infty, b = \infty, K(\omega, t) = e^{-j\omega t}$

2. HANKEL TRANSFORM: $a = 0, b = \infty, K(\lambda, t) = t\,J_{\nu}(\lambda t)$, where $J_{\nu}(\lambda t)$ is Bessel's function (of the first kind) of order $\nu > -1/2$

3. MELLIN TRANSFORM: $a = 0, b = \infty, K(s, t) = t^{s-1}$

4. LAGUERRE TRANSFORM: $a = 0, b = \infty, K(n, t) = e^{-t}L_n(t)$, where $L_n(t)$ is the Laguere polynomial of order n

Such integral transform formulas are usually accompanied by the "inverse" expression of the kind:

$$T^{-1}\{\hat{f}(s)\} = \int_{a^*}^{b^*} K^*(s,t)\,\hat{f}(s)ds = f(t) \tag{C.2}$$

by which the original function, $f(t)$, may be recovered from its transformed version, $\hat{f}(s)$, using $K^*(s, t)$, the kernel of the inverse transform formula, and integrating out the s variable between the limits a^* and b^*. For example, the corresponding inverse Laplace transform expression has:

$a^* = \alpha + j\omega$, $b^* = \alpha - j\omega$ and $K^*(s, t) = e^{st}2\pi j$

and, for the inverse *Hankel transform*:

$a^* = 0$, $b^* = \infty$, and $K^*(\lambda,t) = \lambda\,J_{\nu}(\lambda t)$

The integral transform of Eq. (C.1) and its inverse relation in Eq. (C.2) constitute a "transform pair."

Integral transform pairs in general have found significant application in science and engineering; and some transforms are more natural than others for certain applications. For example, in certain aspects of the theory of elasticity, the Mellin transform is the more natural choice. For process control applications, however, the Laplace and Fourier transforms are the most natural and most useful; and it will be shown later that for the analysis and design of discrete-time control systems, it is in fact just a clever adaptation of the Laplace transform that leads to the more natural z-transform.

C.2 THE FOURIER TRANSFORM

The Fourier series expansion for a function $f(t)$ over the interval $[-L, L]$ is given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \tag{C.3}$$

where the coefficients are given by:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \tag{C.4a}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \tag{C.4b}$$

By substituting Eq. (C.4) into Eq. (C.3) and taking limits as $L \to \infty$, after carefully carrying out the required manipulations, and using the exponential representation for the sin and cos functions, we obtain:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\,d\tau\,d\omega \tag{C.5}$$

This is the Fourier integral formula; it provides the origin of the Fourier transform and its inverse. For, if we were to define:

$$\hat{f}(\omega) \equiv \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{C.6}$$

then Eq. (C.5) immediately becomes:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{j\omega t}\,d\omega \tag{C.7}$$

We may now observe that, in the spirit of the general integral transform pair given in Eqs. (C.1) and (C.2), Eq. (C.6) expresses the transform of the function $f(t)$, and Eq. (C.7) the corresponding inverse. Thus, $\hat{f}(\omega)$ defined as in Eq. (C.6) is called the "Fourier transform" of $f(t)$, and Eq. (C.7) is the corresponding inverse "Fourier transform" formula by which $f(t)$ is recovered from $\hat{f}(\omega)$. Note that the domain of $f(t)$ is infinite.

C.3 THE LAPLACE TRANSFORM

Not all functions are Fourier transformable. In particular, some important functions employed in process dynamics and control — for example, the step function, and the ramp function — are not Fourier transformable because they are functions that do not decay to zero fast enough as $t \to \infty$. Also, the domain of the functions encountered in process dynamics and control is usually the *semi-infinite* region $0 \le t < \infty$ and not the infinite region of the Fourier transform. Thus the Fourier transform may not be the most natural for process dynamics and control problems in general. The Laplace transform is the result of modifying the Fourier transform to better suit such problems.

C.3.1 Definition and Inverse Formula

Consider a function $f(t)$ that satisfies the following conditions:

1. It is "piecewise regular"; (i.e., it is bounded in every interval of the form $0 \le t_1 \le t \le t_2$, and has at most a finite number of maxima and minima, and a finite number of *finite discontinuities*).

2. It is of "exponential order" (i.e., there exist constants a, M, and T such that $e^{-at}|f(t)| < M$ for all $t > T$).

Note that if $e^{-at}|f(t)| < M$ then it is also true that $e^{-a_1t}|f(t)| < M$ for all $a_1 > a$; therefore, the a required by Condition 2 is not unique. Now, let α be the *greatest lower bound* (infimum) of the set of a values for which Condition 2 is satisfied: then α is called the "abscissa of convergence" of the function $f(t)$.

Now, by virtue of Condition 2 and this definition of the abscissa of convergence, for any function satisfying these two conditions, the following composite function:

$$\phi(t) = \begin{cases} 0; & t < 0 \\ e^{-\alpha t} f(t); & t > 0 \end{cases} \tag{C.8}$$

is guaranteed to be Fourier transformable regardless of whether $f(t)$ is or not. Introducing Eq. (C.8) into Eq. (C.5) gives:

$$e^{-\alpha t} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} \int_0^{\infty} (e^{-\alpha\tau} f(\tau))e^{-j\omega\tau}\, d\tau\, d\omega$$

so that:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+j\omega)t} \int_0^{\infty} f(\tau)e^{-(\alpha+j\omega)\tau}\, d\tau\, d\omega \tag{C.9}$$

and by introducing the new variable:

$$s = \alpha + j\omega \tag{C.10}$$

Eq. (C.9) becomes:

$$f(t) = \lim_{\omega\to\infty} \frac{1}{2\pi j}\int_{\alpha-j\omega}^{\alpha+j\omega} e^{st} \int_0^{\infty} f(\tau)e^{-s\tau}\, d\tau\, ds \tag{C.11}$$

and now, by defining:

$$\hat{f}(s) = \int_0^{\infty} f(t)e^{-st}\, dt \tag{C.12}$$

Eq. (C.11) immediately becomes:

$$f(t) = \lim_{\omega \to \infty} \frac{1}{2\pi j} \int_{\alpha - j\omega}^{\alpha + j\omega} \hat{f}(s)\, e^{st}\, ds \tag{C.13}$$

giving the Laplace transform and its inversion formula.

From Eq. (C.10) we may observe that the Laplace transform variable s is complex. Also, the path of integration indicated in the inversion formula is the straight line in the complex plane from $\alpha - j\omega$ to $\alpha + j\omega$ as $\omega \to \infty$; it is known as the *Bromwich path*, and simply denoted by "C" in the expression in Chapter 3. Thus, as initially stated in Chapter 3:

The Laplace transform of a function f(t) that is **piecewise regular** *and of* **exponential order** *is given by Eq. (C.12); the corresponding inverse transform is given by Eq. (C.13).*

In principle, therefore, according to Eq. (C.13), contour integration in the complex plane must be carried out in order to invert a Laplace transform; however, in practice, it is customary to use tables of Laplace transforms, an example of which is presented at the end of this section.

C.3.2 Properties and Useful Theorems of Laplace Transforms

Basic Properties

1. Linearity

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} \tag{C.14}$$

Extension to linear combination of n functions is straightforward.

2. Transform of Derivatives

$$L\left\{\frac{df}{dt}\right\} = s\hat{f}(s) - f(t)\Big|_{t=0} \tag{C.15}$$

$$L\left\{\frac{d^2 f}{dt^2}\right\} = s^2\hat{f}(s) - sf(t)\Big|_{t=0} - \frac{df(t)}{dt}\Big|_{t=0} \tag{C.16}$$

or, in general, if $f^{(n)}(t)$ represents the nth derivative of $f(t)$, and by convention, $f^{(0)}(t) = f(t)$:

$$L\{f^{(n)}(t)\} = s^n\hat{f}(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(t)\Big|_{t=0} \tag{C.17}$$

When all the initial conditions are zero, then:

$$L\{f^{(n)}(t)\} = s^n \hat{f}(s) \tag{C.18}$$

and an nth derivative is turned to an nth power of s. This result is very useful in obtaining transfer functions from differential equation models; it provides the