Zeros in (State) Space!

Douglas A. Allan

Department of Chemical and Biological Engineering

THE UNIVERSITY of WISCONSIN MADISON

February 13, 2017

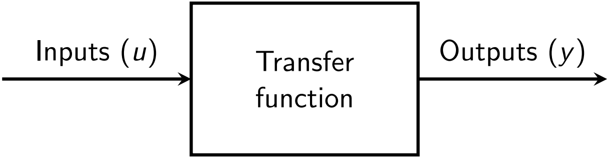
Motivation

- Transfer function zeros are often neglected in undergraduate control
- In single-input/single-output (SISO) systems, there are a few vague descriptions of what they do
  - "Speed up" response
  - Can cause overshoot and inverse responses
- In multiple-input/multiple-output (MIMO) systems, zeros are hard to characterize
  - Poles and zeros can occur at the same place (!)
- It isn't obvious if/where zeros exist in state space models

Outline

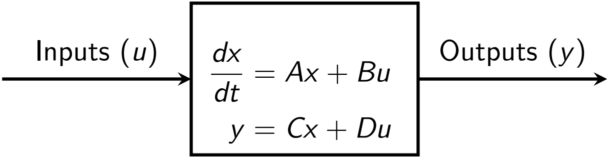
1. What is a transfer function and what is a transfer function zero
2. Properties of zeros in SISO transfer functions
3. How to characterize MIMO transfer function zeros
4. Where zeros go in state-space models

What is a transfer function?



- A transfer function relates process inputs to process outputs
- Explicit transfer functions take several forms
  - Laplace transform
  - Fourier transform
  - Impulse response (Green's function)
- Transfer function is given implicitly by a state space models
- Here we focus on linear Laplace transform transfer functions and state space models

State space models



- System of ODEs that describes evolution of system's internal states
- Outputs are a function of these internal states
- Evolution of outputs with time is obtained by solving the ODEs
- Useful if the inner workings of system are known

Laplace transform

- If inner workings of system are unknown, we can relate the inputs with the outputs directly
- Take Laplace transform of inputs and outputs

$$u(s) = \mathscr{L}(u(t)) = \int_0^\infty u(t)e^{-st}dt \qquad y(s) = \int_0^\infty y(t)e^{-st}dt$$

- Transfer function $G(s)$ satisfies the relationship $y(s) = G(s)u(s)$
  - Transform-domain multiplication corresponds to time-domain convolution
- $G(s)$ can be "realized" into a state-space model, but states do not necessarily correspond to the inner workings of the system
  - If some states are measured, we can force the model to contain them

SISO systems—poles and zeros

- Express transfer function as ratio of polynomials

$$g(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)} = \frac{a_p s^k + a_{p-1} s^{p-1} + \cdots + a_1 s + a_0}{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}$$

- Physically realizable systems have $p \leq q$
- The poles of $g(s)$ are the values of s for which $g(s)$ is singular, i.e., the (possibly complex) roots of $\mathcal{D}(s)$
  - These determine the systems stability— if all poles have a strictly negative real part, the system is stable
- The zeros of $g(s)$ are values of s for which $g(s) = 0$, i.e., the (possibly complex) roots of $\mathcal{N}(s)$
- Note: SISO zeros and poles *cannot* occur at the same value of s

Preliminaries

Initial value theorem

We have that

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} s\mathscr{L}(y)$$

Final value theorem

If all the poles of $g(s)$ have a strictly negative real part, then we have that

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s\mathscr{L}(y)$$

Time domain differentiation

We have that

$$\mathscr{L}\left(\frac{dy}{dt}\right) = s\mathscr{L}(y)$$

Sped up response— Example

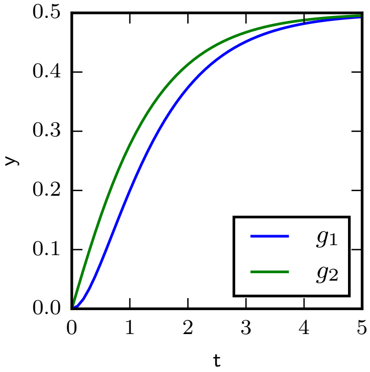
- SISO zeroes "speed up" the response— what does this mean?
- Consider a second order system with no zeros

$$g_1(s) = \frac{1}{(s+1)(s+2)}$$

- System's step response has an initial slope of zero
- Now consider a second order system with a zero

$$g_2(s) = \frac{s+3}{(s+1)(s+2)}$$

s



- This system has a nonzero initial slope

Sped up response— underlying math

- Consider a general qth order system with no zeros

$$g(s) = \frac{C}{(s-\pi_1)(s-\pi_2)\dots(s-\pi_q)}$$

- Find the initial value of the nth derivative of the response of this system to a step input (recall that the Laplace transform of a unit step is $1/s$)

$$\frac{d^n y}{dt^n} = \lim_{s\to\infty}(s)(s^n)g(s)u(s) = \lim_{s\to\infty}\frac{Cs^n}{(s-\pi_1)\dots(s-\pi_q)}$$

  - If $n < q$, then the nth derivative is zero
  - If $n = q$, then the nth derivative attains some nonzero value
  - If $n > q$, then the nth derivative is not well defined (because of the discontinuity of the step function)

Sped up response— underlying math

- Now consider a general qth order system with p zeros

$$g(s) = \frac{C(s-\zeta_1)(s-\zeta_2)\dots(s-\zeta_p)}{(s-\pi_1)(s-\pi_2)\dots(s-\pi_q)}$$

- Repeating the same analysis, we calculate the initial value of the nth derivative

$$\frac{d^n y}{dt^n} = \lim_{s\to\infty} \frac{Cs^n(s-\zeta_1)\dots(s-\zeta_p)}{(s-\pi_1)\dots(s-\pi_q)}$$

  - Now, if $n < q - p$, the derivative is zero
  - If $n = q - p$, the derivative attains some finite value
  - If $n > q - p$, the derivative is undefined

- So zeros speed up the system's response in the sense that they reduce the order of the first non-zero derivative

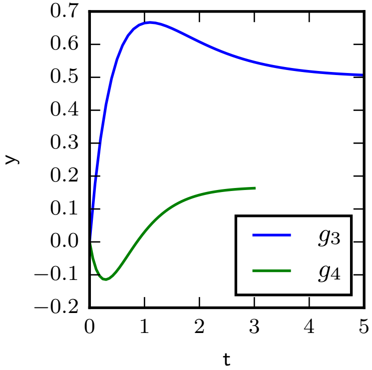
Overshoot and inverse response

- Because they affect the initial response of the system, zeros can cause various phenomena
- Such as overshoot

$$g_3(s) = \frac{2(s + 1/2)}{(s + 1)(s + 2)}$$

- And inverse responses

$$g_4(s) = \frac{1 - s}{(s + 1)(s + 2)}$$



Inverse response— underlying math

- In order to characterize inverse responses, we must determine what the "right" direction for the system to go is
- Consider the unit step response of a stable system

$$\lim_{y\to\infty} y(t) = \lim_{s\to 0} s(g(s))\frac{1}{s} = \lim_{s\to 0} \frac{C(s-\zeta_1)\dots(s-\zeta_p)}{(s-\pi_1)\dots(s-\pi_q)}$$
$$= \frac{C\prod_{i=1}^{p} -\zeta_i}{\prod_{i=1}^{q} -\pi_q} := K$$

- With the system gain K,we define the time constants $\tau_i = -1/\pi_i$ and rewrite the transfer function

$$g(s) = \frac{K\prod_{i=1}^{p} 1 - s/\zeta_i}{\prod_{i=1}^{q} \tau_i s + 1}$$

Inverse response— underlying math

- Now use initial value theorem to find sign of first nonzero derivative

$$\lim_{t \to 0} \frac{d^{q-p}y}{dt^{q-p}} = \lim_{s \to \infty} s^{q-p} g(s) = \lim_{s \to \infty} \frac{K s^{q-p} \prod_{i=1}^{p} 1 - s/\zeta_i}{\prod_{i=1}^{q} (\tau_i s + 1)}$$
$$= \frac{K \prod_{i=1}^{p} -1/\zeta_i}{\prod_{i=1}^{q} \tau_i}$$

- Because the system is stable, $\tau_i > 0$
- If we have an even number of positive zeros, initial response is in the right direction
- If we have an odd number, there is an inverse response

Input blocking

- So far, we've covered stuff that we teach to undergrads in CBE 470
  - Perhaps a bit more math than they could stomach
- These features don't translate well to MIMO systems
- The input-blocking property of zeros does translate well

Input blocking

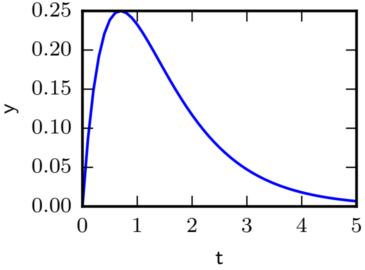
- Consider the transfer function with an inverse response

$$g_4(s) = \frac{1 - s}{(s+2)(s+3)}$$

- What happens when we input a growing exponential?

$$u(t) = e^t \qquad u(s) = \frac{1}{s-1}$$

- A bounded, decaying output...

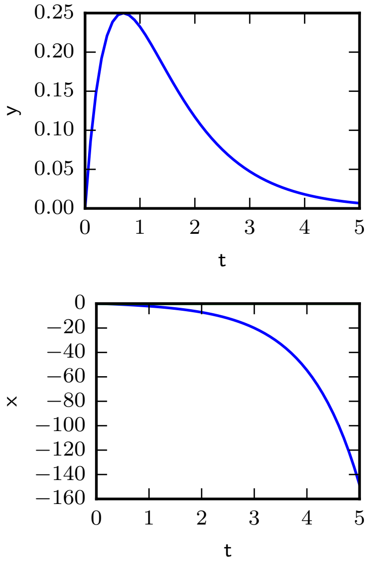


Input blocking

- The process output looks good— let's check in on the process's internal states
- State space realization:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} x$$

- Yikes!



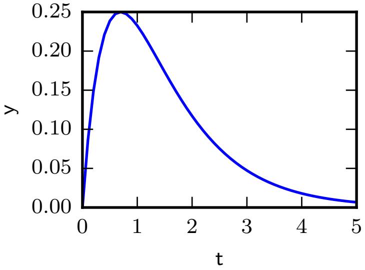
Input blocking

- Because the zero canceled the exponential term, the output converged to zero
- The unobserved state, in contrast, grew exponentially
- Let's look at the effects of a negative zero

$$g(s) = \frac{s+1}{(s+2)(s+3)}$$

- Feed in the corresponding decaying exponential

$$u(t) = e^{-t} \quad u(s) = \frac{1}{s+1}$$



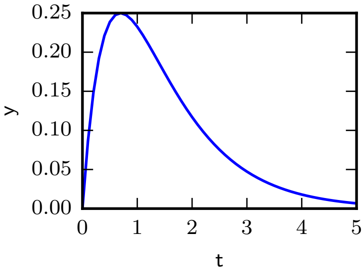
- Like the negative zero, the output decays exponentially

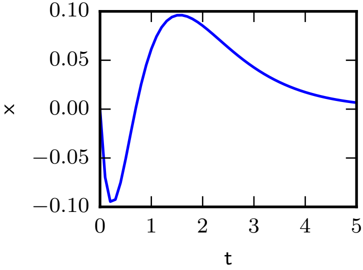
Input blocking

- Now let's look at the hidden state
- State space realization:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

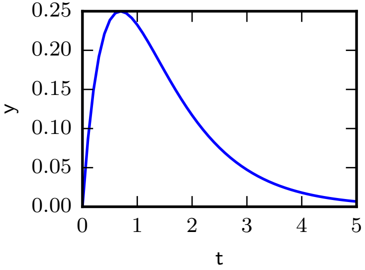
- Somewhat anticlimactic





Input blocking

- So what makes negative zeros special?
  - The state isn't blowing up without our knowledge
  - There's a response in y, so the input isn't blocked completely
- There's a hidden state in our model— what if we give it an initial condition?
  - In particular, set $x_1(0) = -1$
- The exact same response as when we had the input $u = e^{-t}$



Input blocking

- Zeros, regardless of their sign, are special because inputs of those frequencies are indistinguishable from dynamics resulting from hidden states
- Positive zeros are important, however, because the state can blow up without it showing in the outputs
  - Useful to show to practitioners who don't care about state stability as long as the outputs look fine
- This characterization of zeros is readily extensible to MIMO systems

Multi-Input/Multi-Output Systems

- Now instead of having a single transfer function, we have an array

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$

- The poles of this MIMO transfer function are the values of s for which any $g_{ij}(s)$ is singular
- What about zeros? Are they the values for which any $g_{ij}(s)$ is zero?

Attempt 1: Zero elements

- Consider a diagonal transfer function matrix
  - Corresponds to several systems in parallel

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s+1}{(s+2)(s+3)} & 0 \\ 0 & 0 & \frac{1}{s^2+1} \end{bmatrix}$$

- This definition would imply that we have system zeros for all s
- However, for almost all s we do not have any sort of input-blocking property
  - The effects of u_1 may not appear in y_2 and y_3, but they do appear in y_1
- A satisfactory definition of MIMO zeros treats the entire transfer function as a single system, not an array of SISO transfer functions

Attempt 2: Poles of $(G(s))^{-1}$

- For SISO systems, we have that the zeros of $g(s)$ are the poles of $1/g(s)$
- What if we defined the zeros of $G(s)$ to be the poles of $G((s))^{-1}$?

$$(G(s))^{-1} = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & \frac{(s+2)(s+3)}{s+1} & 0 \\ 0 & 0 & s^2+1 \end{bmatrix}$$

- Now we have that $s = -1$ is the only zero of this transfer function
  - For these parallel systems, this definition results in the zeros of the MIMO system to be all the zeros of the individual SISO systems
- Note: $s = -1$ is also a pole of $G(s)$! MIMO systems can have poles and zeros in the same location.

Attempt 3: Locations where $G(s)$ loses rank

- The zeros of $(G(s))^{-1}$ is a promising start, but not all MIMO systems are square
- An entire matrix can be "zero" when it no longer has an inverse, i.e., when it loses rank
  - Extends readily to nonsquare systems
- Let's evaluate $G(s)$ at $s = -1$

$$G(s) = \begin{bmatrix} \infty & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

- How do we evaluate rank with ∞ as a value in the matrix?
  - $\det(G(-1))$ is undefined!

Attempt 4: Zero limiting directions

- Recall definition of matrix null space $\text{null}(A) := \{x \mid Ax = 0\}$
  - $\text{null}(A)$ is nontrivial if $x \neq 0$
- Can define $\text{null}(G(\zeta)) := \{u(s) \mid \lim_{s\to\zeta} G(s)u(s) = 0\}$
- Now we say that $G(s)$ has a zero at $s = \zeta$ if it has a nontrivial ($u(\zeta) \neq 0$) member of its null space at $s = \zeta$
- Applying this definition to our example system, we have that

$$\lim_{s\to-1} G(s)u(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s+1}{(s+2)(s+3)} & 0 \\ 0 & 0 & \frac{1}{s^2+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lim_{s\to-1} \begin{bmatrix} 0 \\ \frac{s+1}{(s+2)(s+3)} \\ 0 \end{bmatrix} = 0$$

- Thus $G(s)$ has a zero at $s = -1$
- This definition, however, is hard to apply

Attempt 5: State space to the rescue!

- We found for SISO systems that there was an initial condition for the hidden states that gave the same response as the input at the zero's frequency
- Equivalently, we found that for the input $u(t) = e^{\zeta t}$ there exists an initial condition $x(0)$ such that $y(t) = 0$ for all $t \geq 0$
- Now we seek to find $u(t) = e^{\zeta t} u_0$, in which u_0 is a constant vector, and $x(0)$ such that $y(t) = 0$
- We first realize the transfer function into state space

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

Attempt 5: State space to the rescue!

- Now, we require

$$0 = \frac{dy}{dt} = C\frac{dx}{dt} + D\frac{du}{dt} = CAx + CBe^{\zeta t}u_0 + \zeta De^{\zeta t}u_0$$

- Suppose $\zeta \notin \text{eig}(A)$. Then we must have $x_p = e^{\zeta t}x_0$ as a particular solution to cancel those terms from $u(t)$
- We then require

$$\begin{aligned}\frac{dx}{dt} = \zeta e^{\zeta t}x_0 &= Ae^{\zeta t}x_0 + Be^{\zeta t}u_0 \\ 0 &= (\zeta I - A)e^{\zeta t}x_0 - Be^{\zeta t}u_0 \\ 0 &= (\zeta I - A)x_0 - Bu_0\end{aligned}$$

- Finally, we require

$$\begin{aligned}y = 0 &= Ce^{\zeta t}x_0 + De^{\zeta t}u_0 \\ &= Cx_0 + Du_0\end{aligned}$$

Attempt 5: State space to the rescue!

- So we have arrived at a set of equations any zero ζ must satisfy

$$0 = \begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$$

- If $\zeta \in \text{eig}(A)$, it must satisfy these equations as well, but this derivation does not work
- If $\zeta \notin \text{eig}(A)$, we have that $\zeta I - A$ is invertible, and therefore

$$x_0 = (\zeta I - A)^{-1} B u_0$$
$$0 = (C(\zeta I - A)^{-1} B + D) u_0$$

- Note that $G(s) = C(sI - A)^{-1}B + D$, so we have found a nontrivial element of the nullspace of $G(s)$

Conclusions

- Although SISO zeros have many properties, the most important one to generalizing them to MIMO systems is the input blocking property
- MIMO zeros are tricky to define correctly— it took decades to finally converge to the right definition
- Neither transfer functions nor state space models are adequate alone to understand zeros; both are necessary

References

- MIT Opencourseware
  - Lecture notes by Frazzoli and Dahleh, Feedback Control Systems, Aeronautics and Astronautics, 2010
  - Lecture notes by Dahleh, Dahleh, and Verghese, Dynamic Systems and Control, Electrical Engineering and Computer Science, 2011
- Ogunnaike and Ray, Process Dynamics, Modeling, and Control, 1994