Markov Process and its Application to Maintenance Optimization

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Outline

- **Markov process**
  - **Discrete-time Markov process**
  - **Continuous-time Markov process**
- **Markov decision process**
- **Application: optimization of maintenance**

Discrete-time Markov Process

Motivating example: Gambler's Ruin

- A gambling game in which on any turn you win \$1 with probability $p = 0.6$, or lose \$1 with probability $1 - p = 0.4$.
- Stop if your fortune reaches \$5 or the casino stops you when your fortune reaches \$0
- X_n = your fortune after n plays

$$\underset{1}{\circlearrowright}\ \mathbf{0} \xleftarrow{.4} \mathbf{1} \underset{.6}{\overset{.4}{\leftrightarrows}} \mathbf{2} \underset{.6}{\overset{.4}{\leftrightarrows}} \mathbf{3} \underset{.6}{\overset{.4}{\leftrightarrows}} \mathbf{4} \xrightarrow[.6]{} \mathbf{5}\ \overset{1}{\circlearrowleft}$$

- The probability if you win at time n

$$\Pr(X_{n+1} = i + 1 | X_n = i) = \Pr(X_{n+1} = i + 1 | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = 0.6$$

- X_n has the *Markov property (memoryless)*

Discrete-time Markov process

X_n is a discrete-time Markov process with transition probability $p(i, j)$ if for any $j, i, i_{n-1}, \ldots, i_0$

$$\Pr(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = Pr(X_{n+1} = j | X_n = i) = p(i, j)$$

Gambler's Ruin Chain

$$\underset{1}{\circlearrowright}\ \mathbf{0} \xleftarrow{.4} \mathbf{1} \underset{.6}{\overset{.4}{\leftrightarrows}} \mathbf{2} \underset{.6}{\overset{.4}{\leftrightarrows}} \mathbf{3} \underset{.6}{\overset{.4}{\leftrightarrows}} \mathbf{4} \xrightarrow[.6]{} \mathbf{5}\ \overset{1}{\circlearrowleft}$$

$p(i, i+1) = 0.6, p(i, i-1) = 0.4$ if $0 < i < 5$

$p(0,0) = 1, p(5,5) = 1$

Transition probability matrix

$$P_{i,j} = p(i,j)$$

$$\sum_j p(i,j) = 1 \ \ \forall i \in S$$

$$P = \begin{array}{c} \begin{array}{cccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array}$$

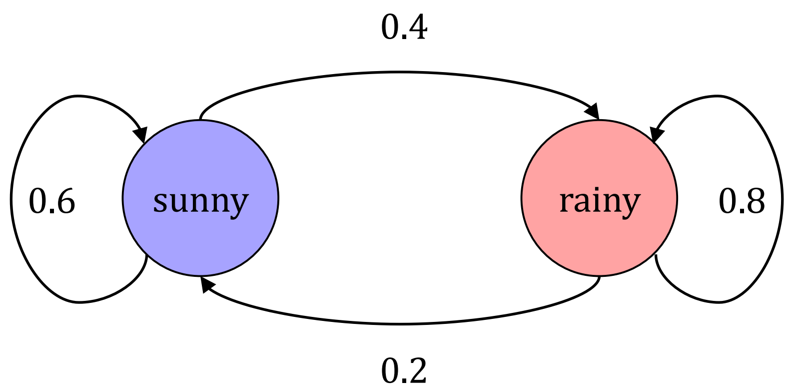
Discrete-time Markov Process

From transition probability to stationary probability

$$\overset{}{\underset{1}{\circlearrowright}}\ \mathbf{0}\ \overset{.4}{\longleftarrow}\ \mathbf{1}\ \underset{.6}{\overset{.4}{\leftrightarrows}}\ \mathbf{2}\ \underset{.6}{\overset{.4}{\leftrightarrows}}\ \mathbf{3}\ \underset{.6}{\overset{.4}{\leftrightarrows}}\ \mathbf{4}\ \underset{.6}{\longrightarrow}\ \mathbf{5}\ \overset{1}{\circlearrowleft}$$

- Q: What is the probability that states can remains unchanged as time progresses?
- A simpler example (weather chain):

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$



Suppose the distribution of state π can remain unchanged after one time step

$$\pi^{\mathrm{T}} P = \pi^{\mathrm{T}}$$

$$[\pi_s \quad \pi_r] \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = [\pi_s \quad \pi_r]$$

$$\pi = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

Continuous-time Markov Process

Why we need continuous-time Markov Process?

- Discrete-time MP: transit at discrete time points
- Continuous-time MP: transit at any time → can better model processes that have continuous dynamics

Examples

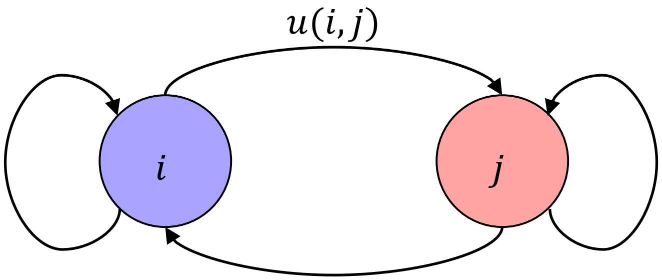
When modeling the waiting time between successive changes

- Reliability engineering: occurrence of failures
- Genetics: DNA mutations
- Ecology: mortality of populations

Continuous-time Markov Process

From discrete-time MP to continuous-time MP

- Example: a two-state discrete-time MP



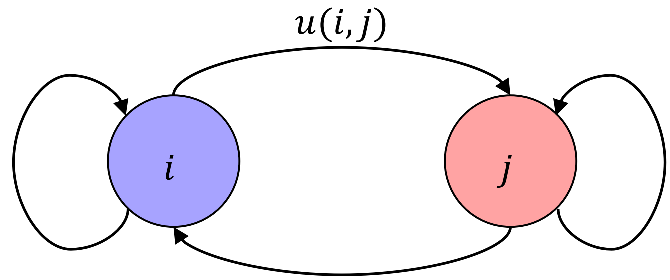
  - The waiting time that the jump from i to j occurs (random variable T) has an exponential distribution with rate λ (i.e.,$\Pr(T = t) = \lambda e^{-\lambda t}$, $\mathbb{E}[T] = 1/\lambda$)

    Memoryless: $\Pr(T > t + s | T > t) = \Pr(T > s)$

Continuous-time Markov Process

From discrete-time MP to continuous-time MP

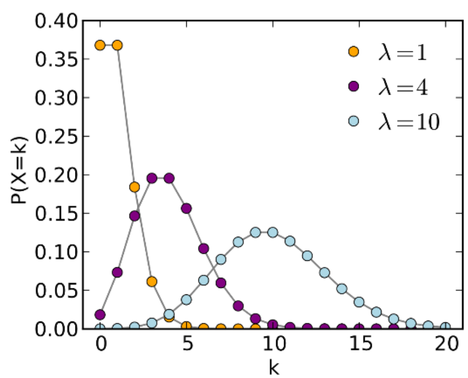
- Example: a two-state discrete-time MP



- The waiting time that the jump from i to j occurs (random variable T) has an exponential distribution with rate λ (i.e.,$\Pr(T = t) = \lambda e^{-\lambda t}, \mathbb{E}[T] = 1/\lambda$)

  Memoryless: $\Pr(T > t + s | T > t) = \Pr(T > s)$

- At any given time t, the number of jump from time 0 to t (random variable $N(t)$) has an Poisson distribution with mean λt (i.e., $\Pr(N(t) = n) = e^{\lambda t} \frac{(\lambda t)^n}{n!}$)



PDF of Poisson distribution. Reprinted from Poisson distribution in *Wikipedia*.

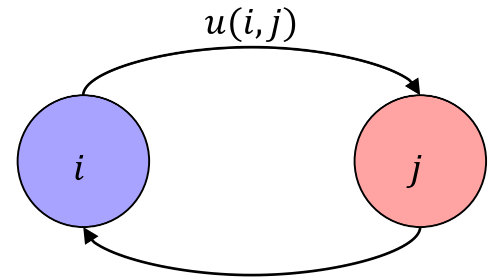
Poisson distribution

$$\Pr(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

Continuous-time Markov Process

From discrete-time MP to continuous-time MP

- Example: a two-state discrete-time MP



  - The waiting time that the jump from i to j occurs (random variable T) has an exponential distribution with rate λ (i.e.,$\Pr(T = t) = \lambda e^{-\lambda t}, \mathbb{E}[T] = 1/\lambda$)

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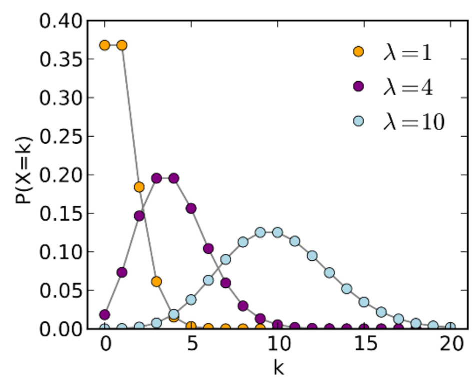


PDF of Poisson distribution. Reprinted from Poisson distribution in *Wikipedia*.

Poisson distribution

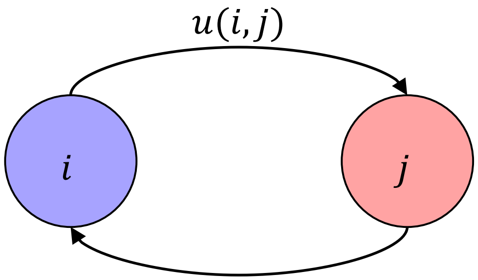
$$\Pr(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

- Q: What's the probability of $p_t(i,j) = P(X_t = j | X_0 = i)$?

Continuous-time Markov Process

From discrete-time MP to continuous-time MP

- Example: a two-state process



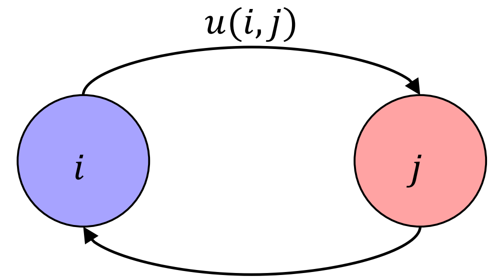
- Q: What's the probability of $p_t(i,j) = \Pr(X_t = j | X_0 = i)$?

$$p_t(i,j) = P(X_t = j | X_0 = i) = \sum_{n=1}^{\infty} \Pr(N(t) = n)$$

Continuous-time Markov Process

From discrete-time MP to continuous-time MP

- Example: a two-state process



- Q: What's the probability of $p_t(i,j) = \Pr(X_t = j | X_0 = i)$?

$$p_t(i,j) = P(X_t = j | X_0 = i) = \sum_{n=1}^{\infty} \Pr(N(t) = n)$$

$$\Pr\left(X_{t+s} = j \middle| X_s = i, X_{s_n} = i_n, \dots, X_{s_0} = i_0\right) = P(X_t = j | X_0 = i)$$

Continuous-time MP

X_t is a continuous-time Markov process with transition probability $p_t(i,j)$ if for any $0 \le s_0 < s_1 < \cdots < s_n < s$

$$\Pr\left(X_{t+s} = j \middle| X_s = i, X_{s_n} = i_n, \dots, X_{s_0} = i_0\right) = \Pr(X_t = j | X_0 = i) = p_t(i,j)$$

Transition rate and transition rate matrix

$$q(i,j) = \lim_{t \to 0} \frac{p_t(i,j)}{t} \; \forall i \neq j$$

How to understand? $\left(\frac{p_t(i,j)}{t} \approx \frac{\mathbb{E}[\text{Number of jumps}]}{\text{Time}}\right)$

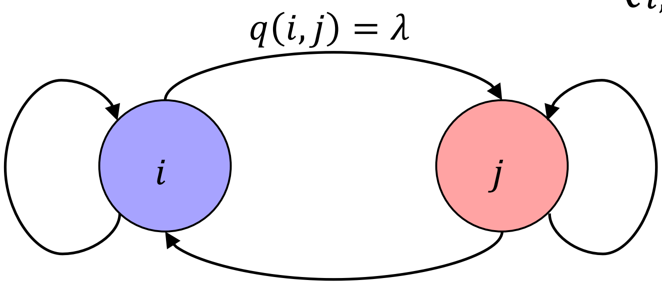
For the example above: $q(i,j) \approx \lambda$

Continuous-time Markov Process

Transition rate and transition rate matrix

$$q(i,j) = \lim_{t \to 0} \frac{p_t(i,j)}{t} \; \forall i \neq j$$

$$Q_{i,j} = \begin{cases} q(i,j) & \text{if } j \neq i \\ -\lambda_i & \text{if } j = i \end{cases}$$



$$Q = \begin{bmatrix} -q(i,j) & q(i,j) \\ q(j,i) & -q(j,i) \end{bmatrix} = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

: transition rate leaving the state

: rate into the state

Stationary distribution

$$\pi^{\mathrm{T}} Q = 0$$

$$[\pi_i \quad \pi_j] \begin{bmatrix} -q(i,j) & q(i,j) \\ q(j,i) & -q(j,i) \end{bmatrix} =$$

$$\left[\pi_i\big(-q(i,j)\big) + \pi_j q(j,i), \quad \pi_i q(i,j) + \pi_j(-q(j,i))\right] = 0$$

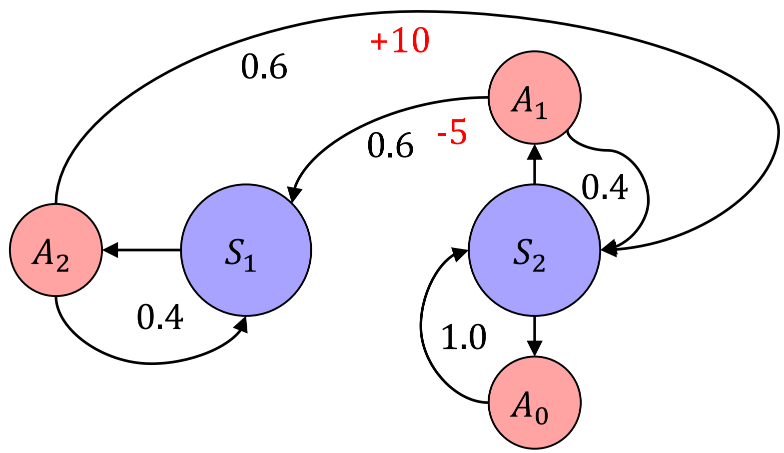
Markov Decision Process

What is a Markov decision process (MDP)?

- A stochastic control process
- Partly random and partly under the control of a decision maker
- Described by a tuple

$$(S, A, P_a, R_a)$$

$P_a(s, s') = P(s_{t+1} = s | s_t = s, a_t = a)$ $R_a(s, s')$



Policy

- Policy: a function Π(s) that maps each state to an action
- Once a deterministic policy is selected, a MDP reduces to a MP
- Goal: find a policy that maximizes the long-term reward $G_t = \sum_{t=0}^{\infty} \gamma^t R_{a_t}(s_t, s_{t+1})$

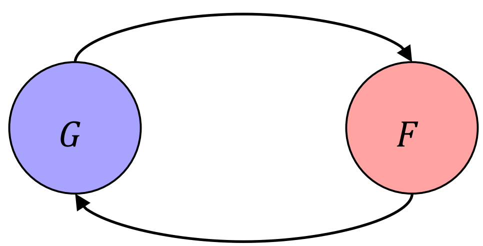
Application to Maintenance

Reliability of Unit

- Reflected by its failures
- Can be examined by its life time (i.e., the time between the consecutive failures)
- The lifetime T has a exponential distribution with rate λ (i.e., the mean life time is $1/\lambda$)

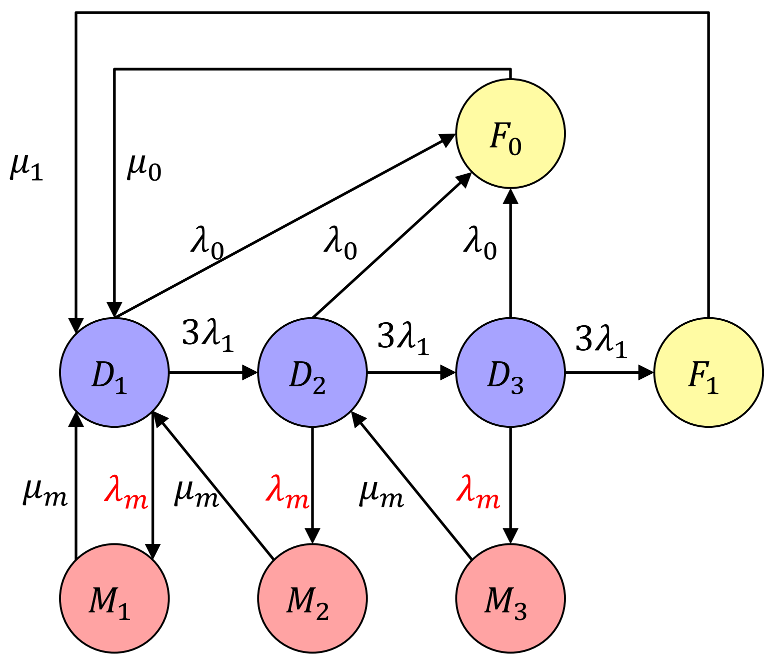
$$\Pr(T \geq t) = e^{\lambda t}$$
$$f_T(t) = \lambda e^{-\lambda t} \; (t \geq 0)$$

$q(G, F) = \lambda$



Application to Maintenance: Case Study

Maximize the availability of the system



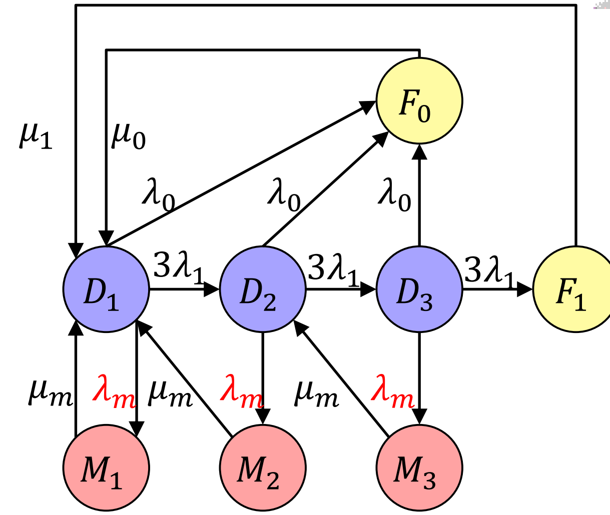
- λ: transition rate related to deterioration, failure or the time to maintenance
- μ: transition rate related to processing time of maintenance time
- Availability= $\pi_{D_1}+\pi_{D_2}+\pi_{D_3}$

Application to Maintenance: Case Study

Maximize the availability of the system

- Availability$= \pi_{D_1} + \pi_{D_2} + \pi_{D_3}$
- Recall $\pi^{\mathrm{T}} Q = 0$
- $$Q = \begin{array}{cc} & \begin{matrix} \boldsymbol{D_1} & \boldsymbol{D_2} & \boldsymbol{D_3} & \boldsymbol{M_1} & \boldsymbol{M_2} & \boldsymbol{M_3} & \boldsymbol{F_0} & \boldsymbol{F_1} \end{matrix} & \\ & \begin{bmatrix} -\Sigma_1 & 3\lambda_1 & & \lambda_m & & & \lambda_0 & \\ & -\Sigma_2 & 3\lambda_1 & & \lambda_m & & \lambda_0 & \\ & & -\Sigma_3 & & & \lambda_m & \lambda_0 & 3\lambda_1 \\ \mu_m & & & -\mu_m & & & & \\ \mu_m & & & & -\mu_m & & & \\ & \mu_m & & & & -\mu_m & & \\ \mu_0 & & & & & & -\mu_0 & \\ \mu_1 & & & & & & & -\mu_1 \end{bmatrix} & \begin{matrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \\ \boldsymbol{D_3} \\ \boldsymbol{M_1} \\ \boldsymbol{M_2} \\ \boldsymbol{M_3} \\ \boldsymbol{F_0} \\ \boldsymbol{F_1} \end{matrix} \end{array}$$
- $\pi^{\mathrm{T}} e = 1$
- Availability$(\lambda_m) = \pi_{D_1} + \pi_{D_2} + \pi_{D_3}$
- If $\lambda_0 = 500, \lambda_m \approx \frac{1}{203}$

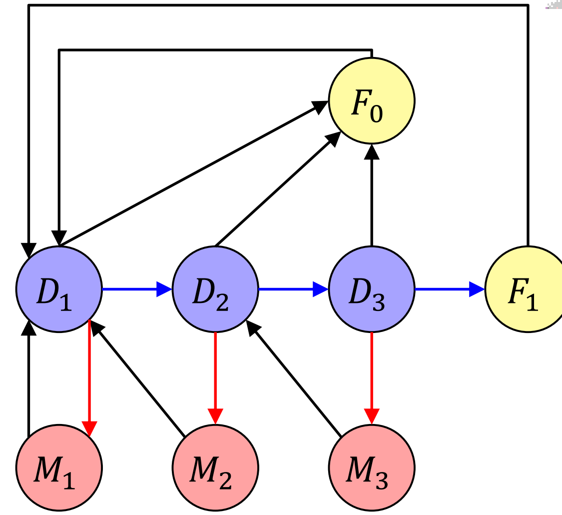
  Availability$(\lambda_m) \approx 0.98$



Application to Maintenance: Case Study

Find the optimal maintenance policy

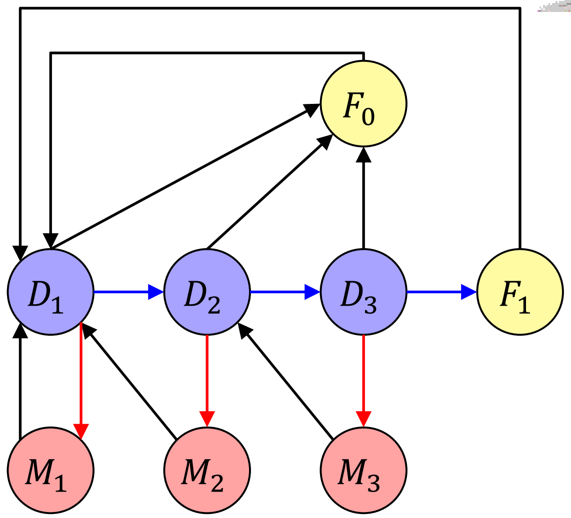
- At each deterioration state, we have two actions to choose
  - Do maintenance
  - Do nothing
- Q: what action to choose to maximize the reward?



Reward	Transition probabilities
$r(D_1, I, F_0) = -500$	$P(F_0 \mid D_1, I) = 0.1349$
$r(D_1, I, D_2) = 900$	$P(D_2 \mid D_1, I) = 0.1978$
$r(D_1, I, D_1) = 1000$	$P(D_1 \mid D_1, I) = 0.6703$
$r(D_1, II, M_1) = -100$	$P(M_1 \mid D_1, II) = 1.0$
$r(D_2, I, D_3) = 800$	$P(D_3 \mid D_2, I) = 0.4570$
$r(D_2, I, F_0) = -500$	$P(F_0 \mid D_2, I) = 0.3046$
$r(D_2, I, D_2) = 900$	$P(D_2 \mid D_2, I) = 0.2384$
$r(D_2, II, M_2) = -100$	$P(M_2 \mid D_2, II) = 1.0$
$r(D_3, I, F_1) = -1000$	$P(F_1 \mid D_3, I) = 0.5567$
$r(D_3, I, D_3) = 800$	$P(D_3 \mid D_3, I) = 0.0721$
$r(D_3, I, F_0) = -500$	$P(F_0 \mid D_3, I) = 0.3712$
$r(D_3, II, M_3) = -100$	$P(M_3 \mid D_3, II) = 1.0$

Application to Maintenance: Case Study

Find the optimal maintenance policy



- At each deterioration state, we have two actions to choose
  - Do maintenance
  - Do nothing
- Q: what action to choose to maximize the long-term reward?
- Using policy iteration we get the optimal policy:

State	D_1	D_2	D_3
Action	Do nothing	Do maintenance	Do maintenance

References

- Durrett, R. (2017). *Essentials of stochastic processes*. Springer.
- Chan, G. K., & Asgarpoor, S. (2006). Optimum maintenance policy with Markov processes. *Electric power systems research*, *76*(6-7), 452-456.
- Ye, Y., Grossmann, I. E., Pinto, J. M., & Ramaswamy, S. (2019). Modeling for Reliability Optimization of System Design and Maintenance Based on Markov Chain Theory. *Computers & Chemical Engineering*.