The Pooling Problem

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Computing Seminar
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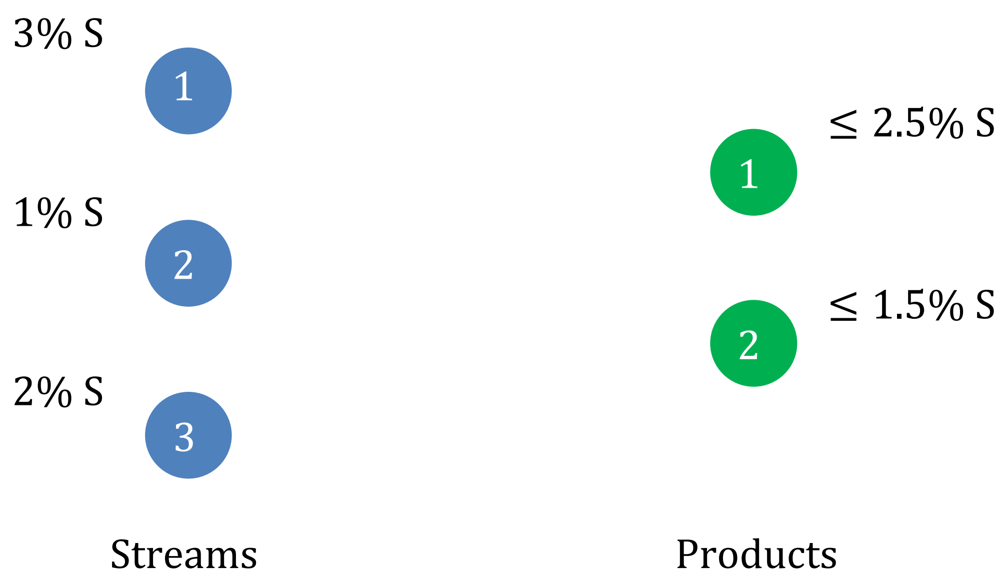
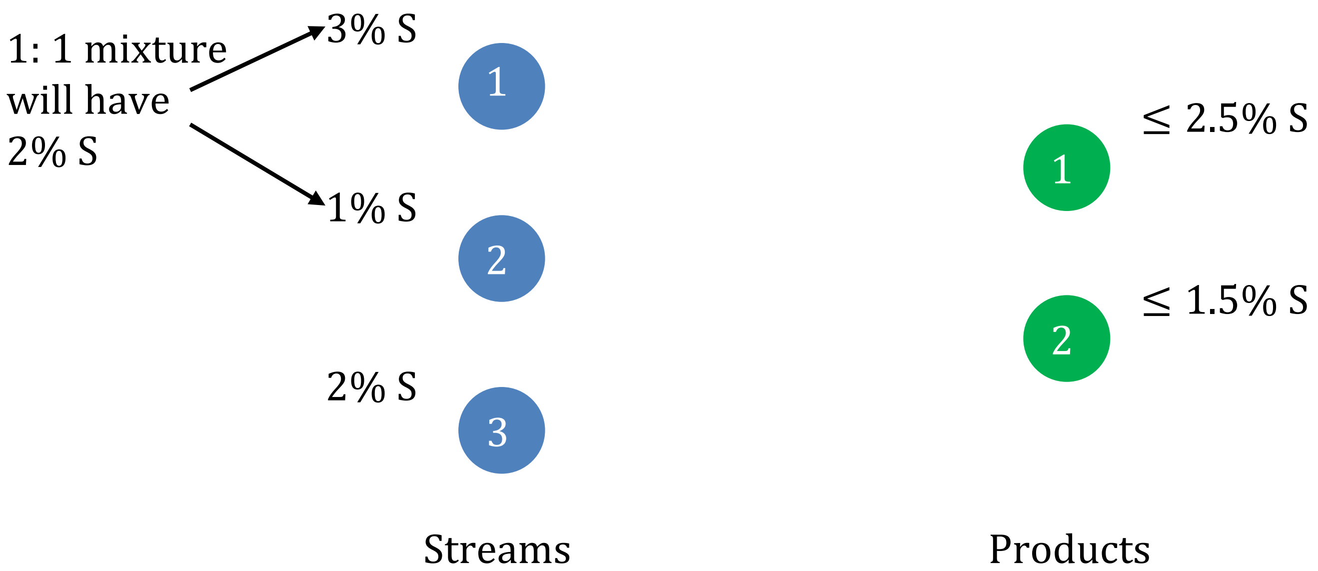
Introduction

- Mixing *streams* with different *property* is common in many industries
- The mixtures (final *products*) must satisfy certain *specifications*



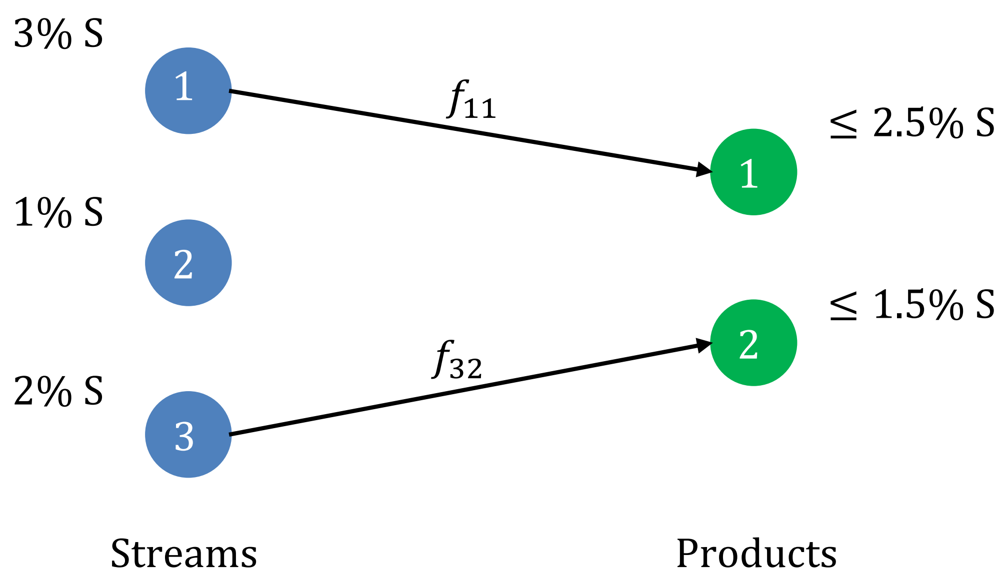
Introduction

- Mixing *streams* with different *property* is common in many industries
- The mixtures (final *products*) must satisfy certain *specifications*
- We assume linear mixing, that is, property can be calculated by a volumetric average



Blending without Pool: An LP Problem

f : Flow variable for streams to products (6 in total)



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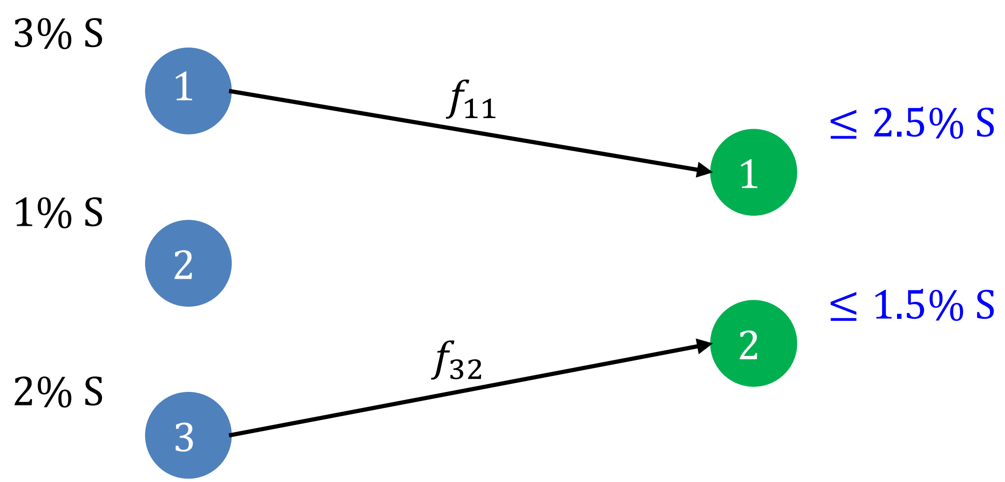

$$3f_{11} + f_{21} + f_{31} \le 2.5(f_{11} + f_{21} + f_{31})$$
$$3f_{12} + f_{22} + f_{32} \le 2.5(f_{12} + f_{22} + f_{32})$$

Specification

Blending without Pool: An LP Problem

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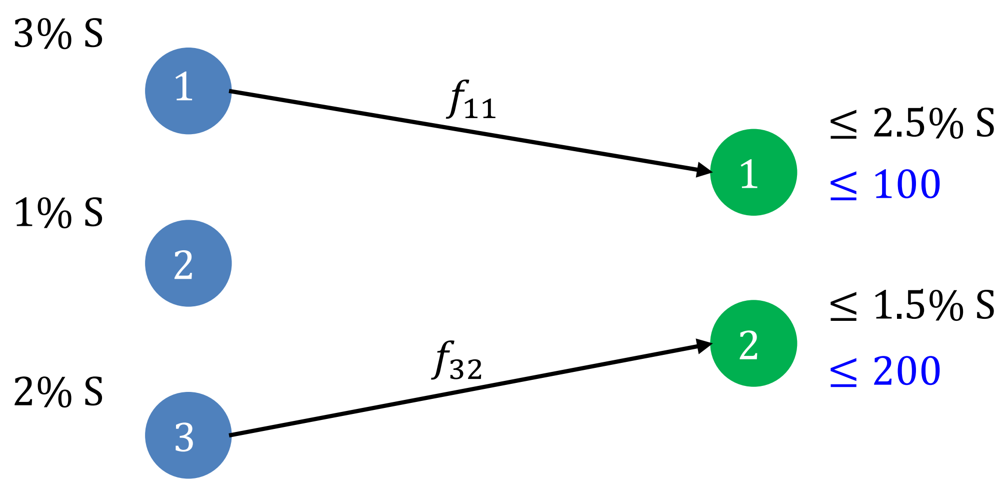

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Specification

$$f_{11} + f_{21} + f_{31} \leq 100$$
$$f_{12} + f_{22} + f_{32} \leq 200$$

Max. demand

Blending without Pool: An LP Problem

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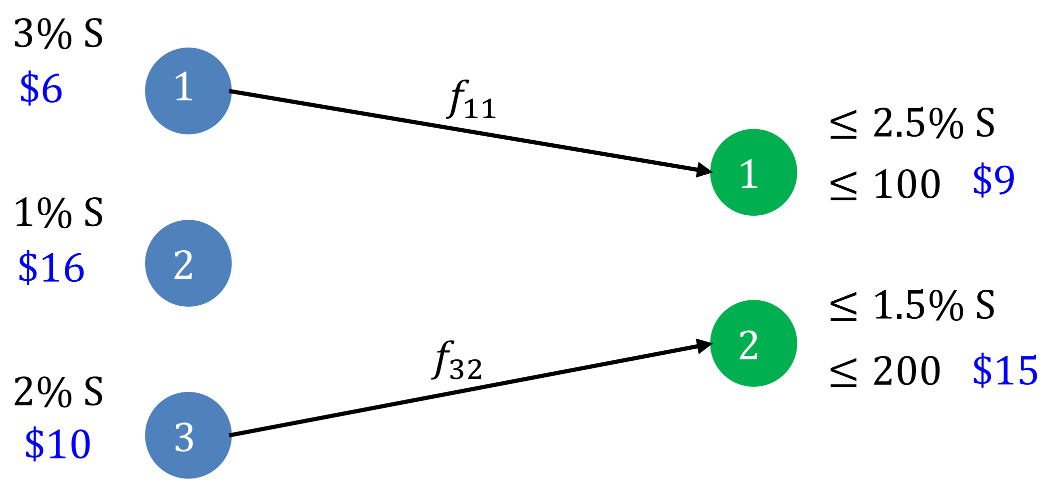

Objective: max $(Revenue - Cost)$

$$Revenue = 9(f_{11} + f_{21} + f_{31}) + 15(f_{12} + f_{22} + f_{32})$$

$$Cost = 6(f_{11} + f_{12}) + 16(f_{21} + f_{22}) + 10(f_{31} + f_{32})$$

Blending without Pool: An LP Problem

$$\max \quad (Revenue - Cost)$$

s. t.

$$Revenue = 9(f_{11} + f_{21} + f_{31}) + 15(f_{12} + f_{22} + f_{32})$$

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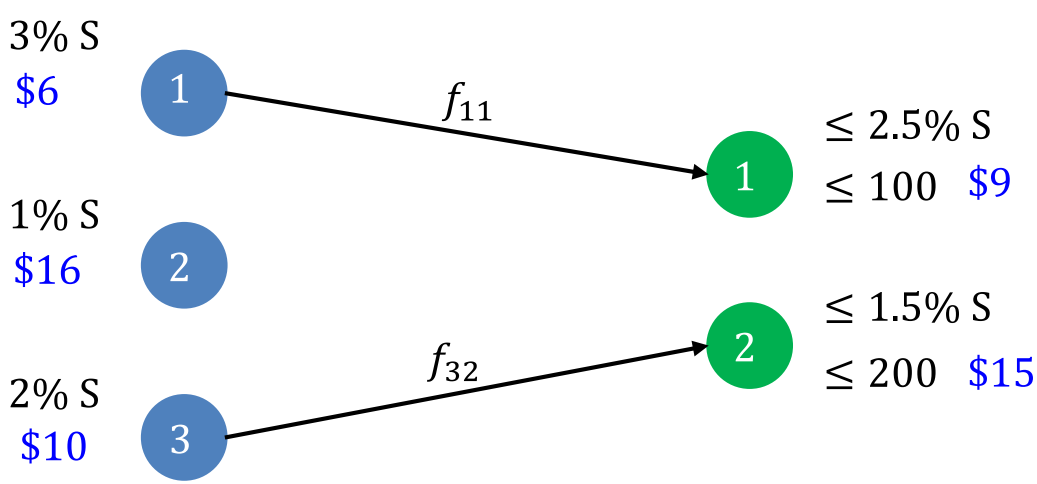
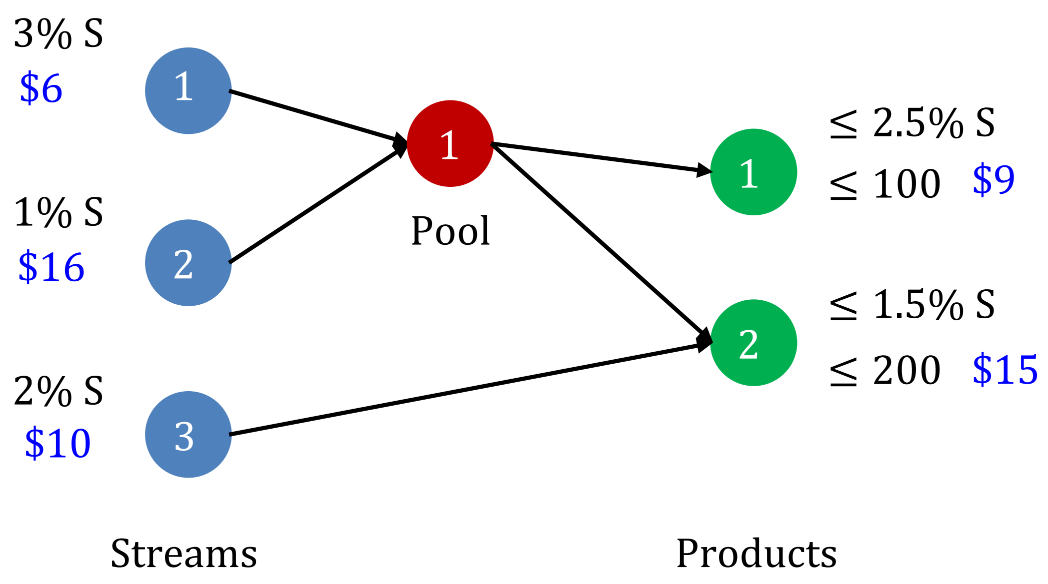
Linear Programming



The Pooling Problem

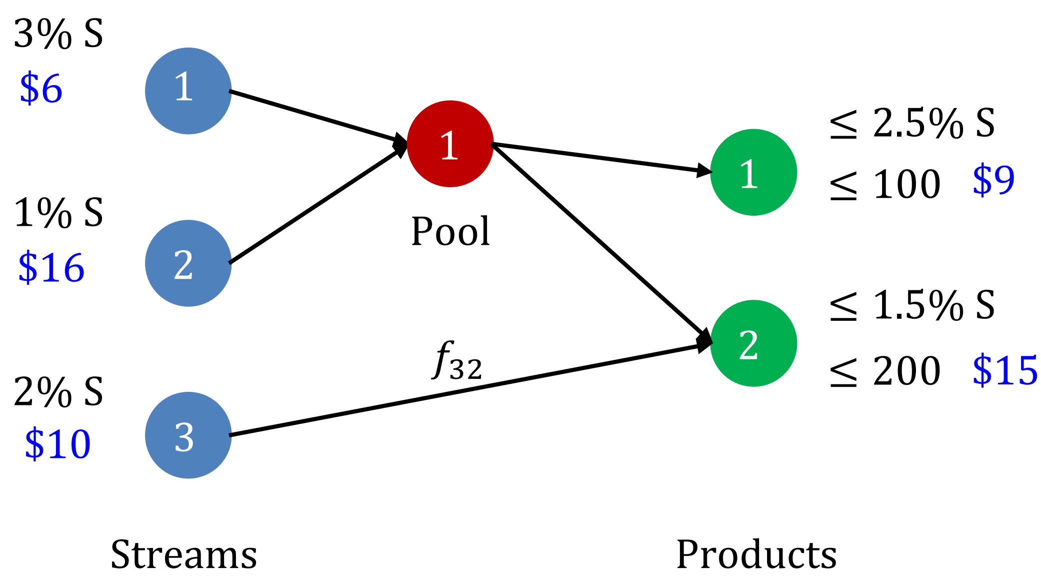
Streams may have to be pooled together due to limited storage unit



Haverly 1978

The Pooling Problem: *q*-Formulation

f : Flow variable for streams to products (2 in total)



Ben-Tal et al.,1994

The Pooling Problem: *q*-Formulation

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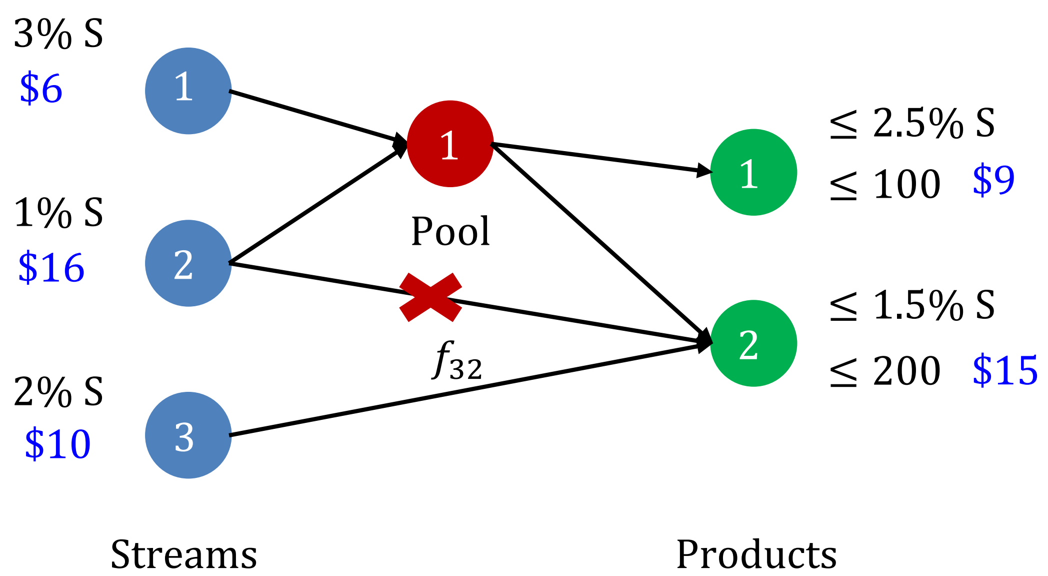
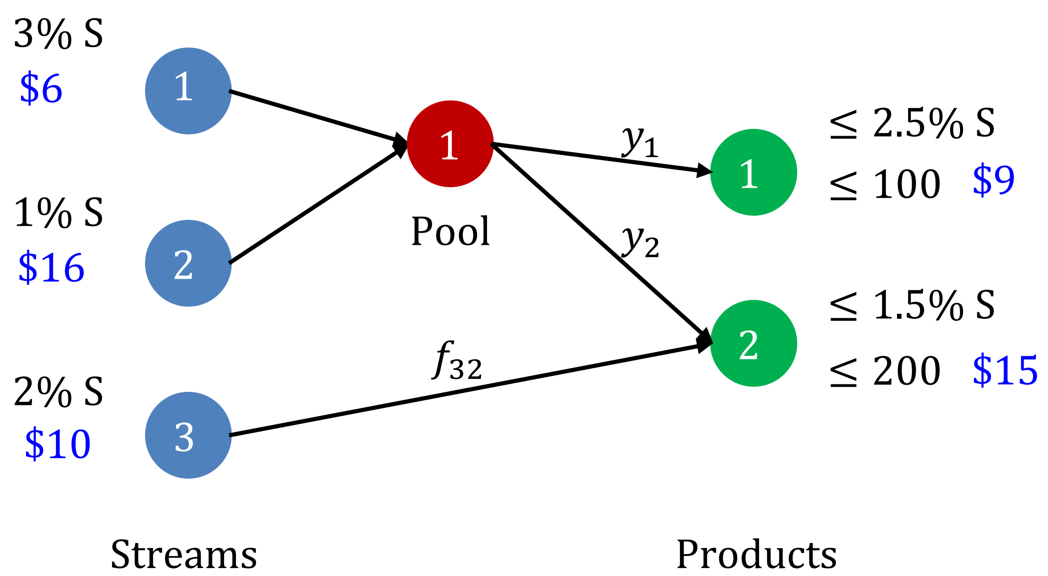


Ben-Tal et al.,1994

The Pooling Problem: *q*-Formulation

f : Flow variable for streams to products (2 in total)

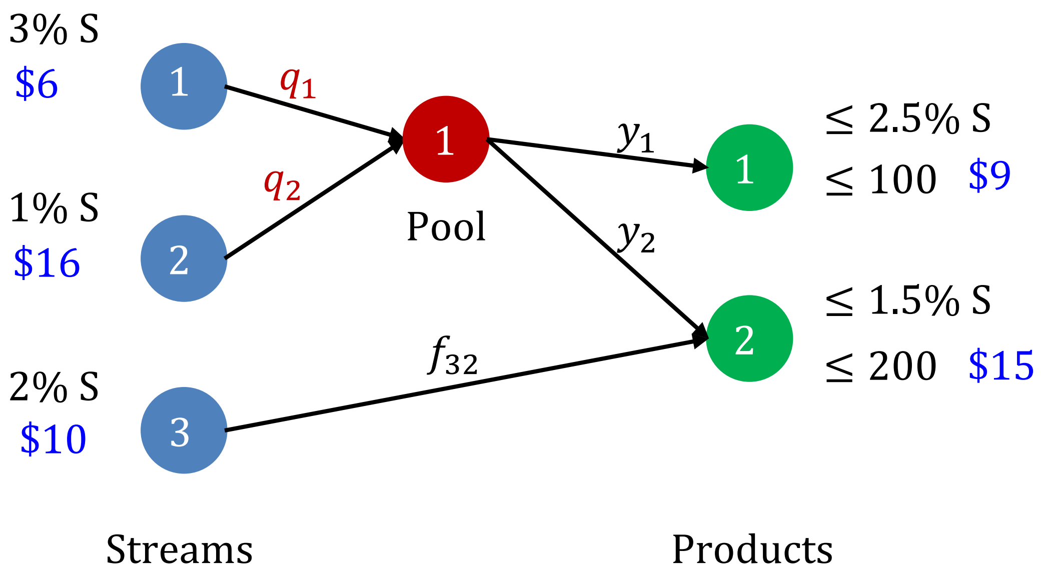
y : Flow variable for pool to products (2 in total)



Ben-Tal et al.,1994

The Pooling Problem: *q*-Formulation

f : Flow variable for streams to products (2 in total)
y : Flow variable for pool to products (2 in total)
q : Stream fraction in pool (2 in total)



$q_1 + q_2 = 1$

Ben-Tal et al.,1994

The Pooling Problem: *q*-Formulation

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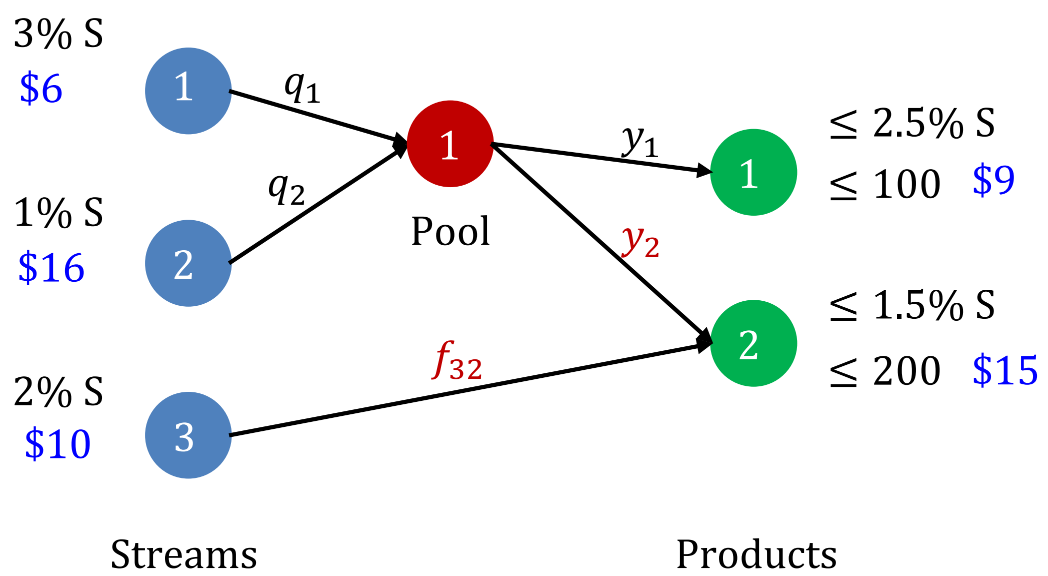

$$y_1 + f_{31} \leq 100$$
$$y_2 + f_{32} \leq 200$$

Max. Demand

Ben-Tal et al.,1994

The Pooling Problem: q-Formulation

f : Flow variable for streams to products (2 in total)
y : Flow variable for pool to products (2 in total)
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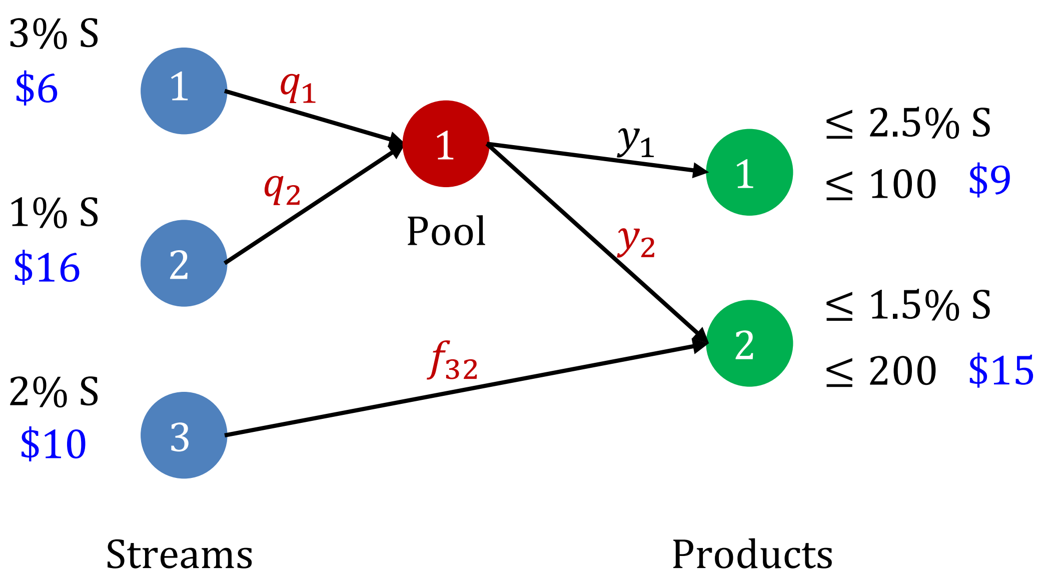

$$3y_1q_1 + y_1q_2 + 2f_{31} \leq 2.5(y_1 + f_{31})$$

$$3y_2q_1 + y_2q_2 + 2f_{32} \leq 1.5(y_1 + f_{31})$$ Specification

Ben-Tal et al.,1994

The Pooling Problem: *q*-Formulation

f : Flow variable for streams to products (2 in total)

y : Flow variable for pool to products (2 in total)

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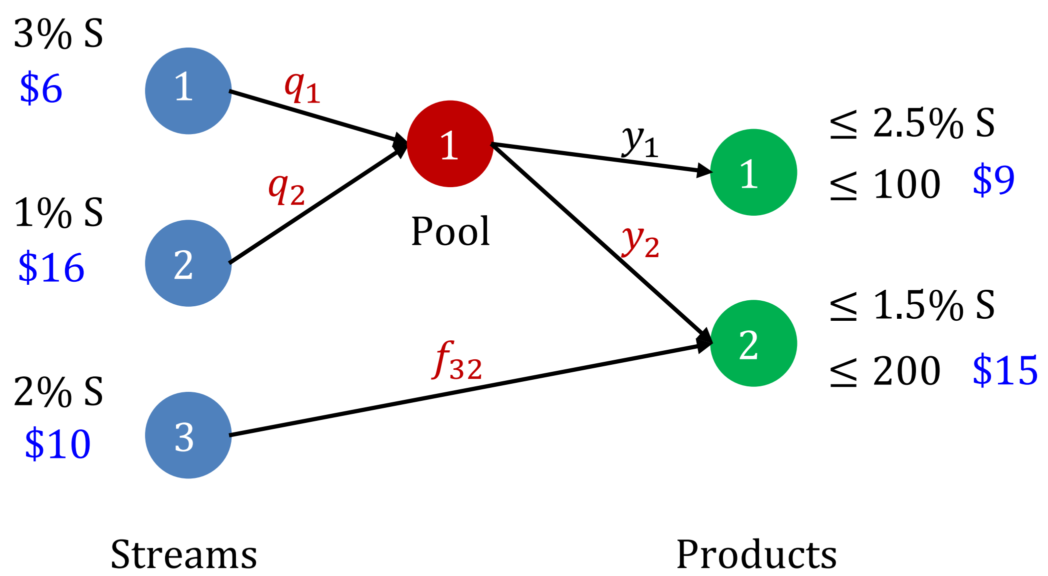

Revenue from product 2: $15(y_2 + f_{32})$

Cost from stream 1: $6q_1(y_1 + y_2)$

Ben-Tal et al.,1994

The Pooling Problem: q-Formulation

$$\max \quad (Revenue - Cost)$$

$$\text{s.t.}$$

$$Revenue = 9(y_1 + f_{31}) + 15(y_2 + f_{32})$$

$$Cost = 6q_1(y_1 + y_2) + 16q_2(y_1 + y_2) + 10(f_{31} + f_{32})$$

$$q_1 + q_2 = 1$$

$$y_1 + f_{31} \leq 100$$

$$y_2 + f_{32} \leq 200$$

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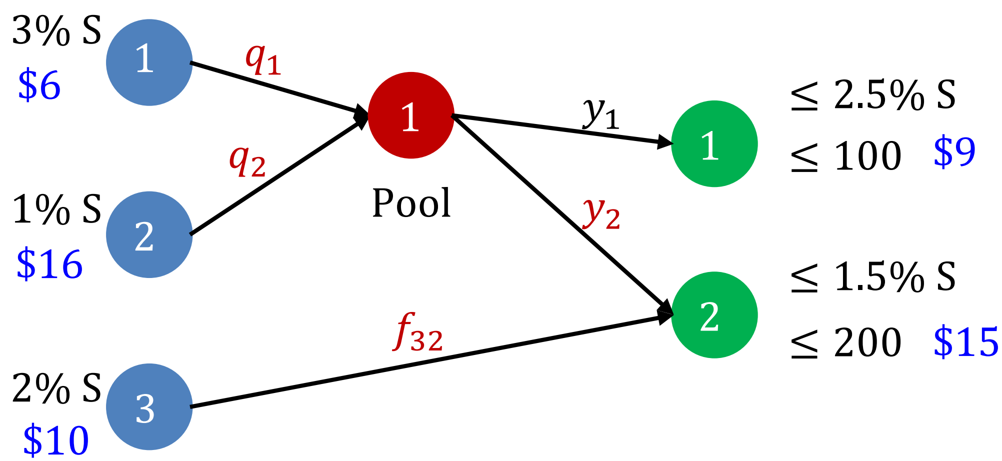


(*Nonconvex*) Nonlinear Programming

Much more difficult to solve (to global optimal)

More on Computing

Nonconvex → Local solution may not be global optimal

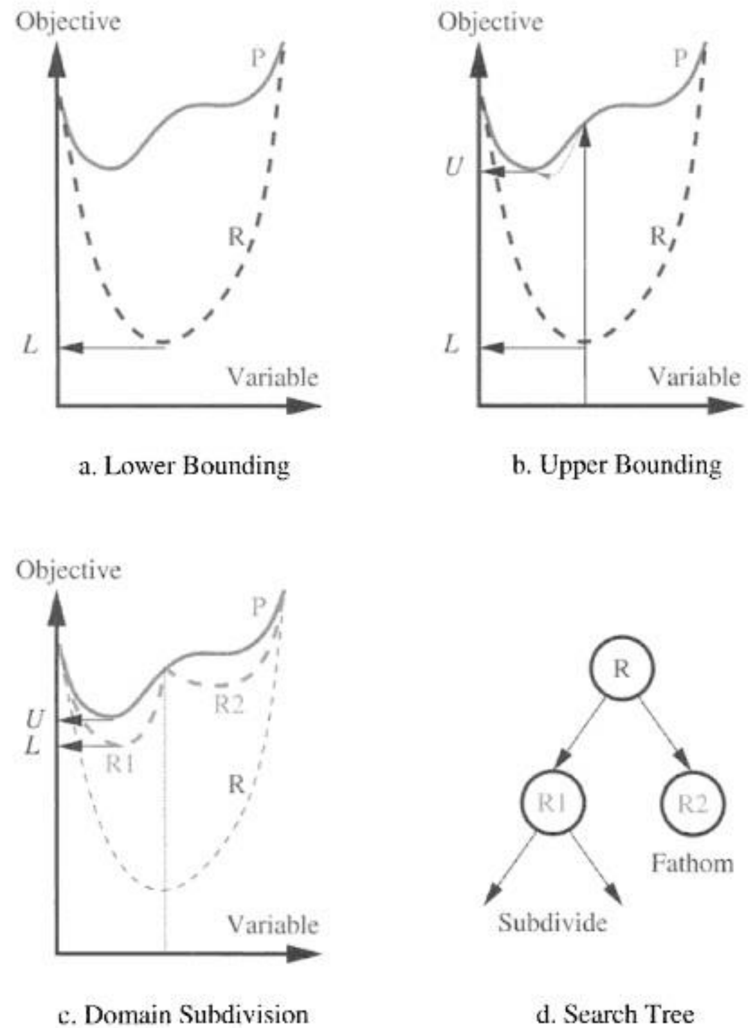


Figure 1.4: The principles of branch-and-bound

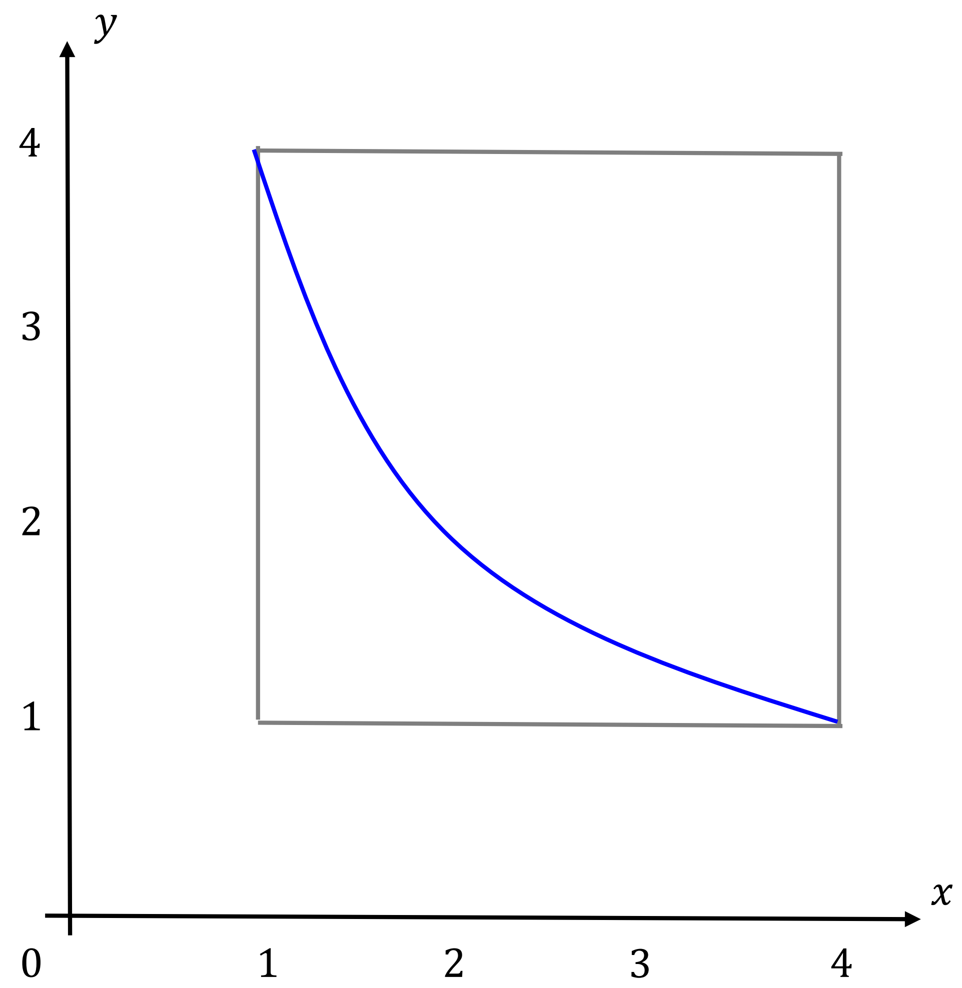
Tawarmalani and Sahinidis, 2002

B&B Example

$$\min \quad x + y$$

$$\text{s.t.} \quad xy = 4, \quad x \in [1,4], \quad y \in [1,4]$$

Feasible space:
blue curve



B&B Example

$$\min \quad x + y$$

$$\text{s.t.} \quad \begin{aligned} xy &= 4 \\ x &\in [1,4] \\ y &\in [1,4] \end{aligned} \qquad x^{\mathrm{L}} = 1, x^{\mathrm{U}} = 4$$

Step 1. Lower bounding

We replace $xy = 4$ with the following:

$$x^{\mathrm{L}}y + xy^{\mathrm{L}} - x^{\mathrm{L}}y^{\mathrm{L}} \leq 4$$
$$x^{\mathrm{U}}y + xy^{\mathrm{U}} - x^{\mathrm{U}}y^{\mathrm{U}} \leq 4$$
$$x^{\mathrm{U}}y + xy^{\mathrm{L}} - x^{\mathrm{U}}y^{\mathrm{L}} \geq 4$$
$$x^{\mathrm{L}}y + xy^{\mathrm{U}} - x^{\mathrm{L}}y^{\mathrm{U}} \geq 4$$

McCormick, 1976

B&B Example

$$\min \quad x + y$$

$$\text{s.t.} \quad xy = 4, \quad x \in [1,4], \quad y \in [1,4]$$

$x^{\mathrm{L}} = 1, x^{\mathrm{U}} = 4$

Step 1. Lower bounding

We replace $xy = 4$ with the following:

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$x^{\mathrm{U}}y + xy^{\mathrm{L}} - x^{\mathrm{U}}y^{\mathrm{L}} \geq 4$
$x^{\mathrm{L}}y + xy^{\mathrm{U}} - x^{\mathrm{L}}y^{\mathrm{U}} \geq 4$

McCormick, 1976

Consider
$x^{\mathrm{U}} - x \geq 0$
$y - y^{\mathrm{L}} \geq 0$
Thus
$(x^{\mathrm{U}} - x)(y - y^{\mathrm{L}}) \geq 0$

$x^{\mathrm{U}}y - x^{\mathrm{U}}y^{\mathrm{L}} - xy + xy^{\mathrm{L}} \geq 0$
$x^{\mathrm{U}}y - x^{\mathrm{U}}y^{\mathrm{L}} + xy^{\mathrm{L}} \geq xy = 4$

$x^{\mathrm{U}}y - x^{\mathrm{U}}y^{\mathrm{L}} + xy^{\mathrm{L}} \geq 4$

B&B Example

$$\min \quad x + y$$

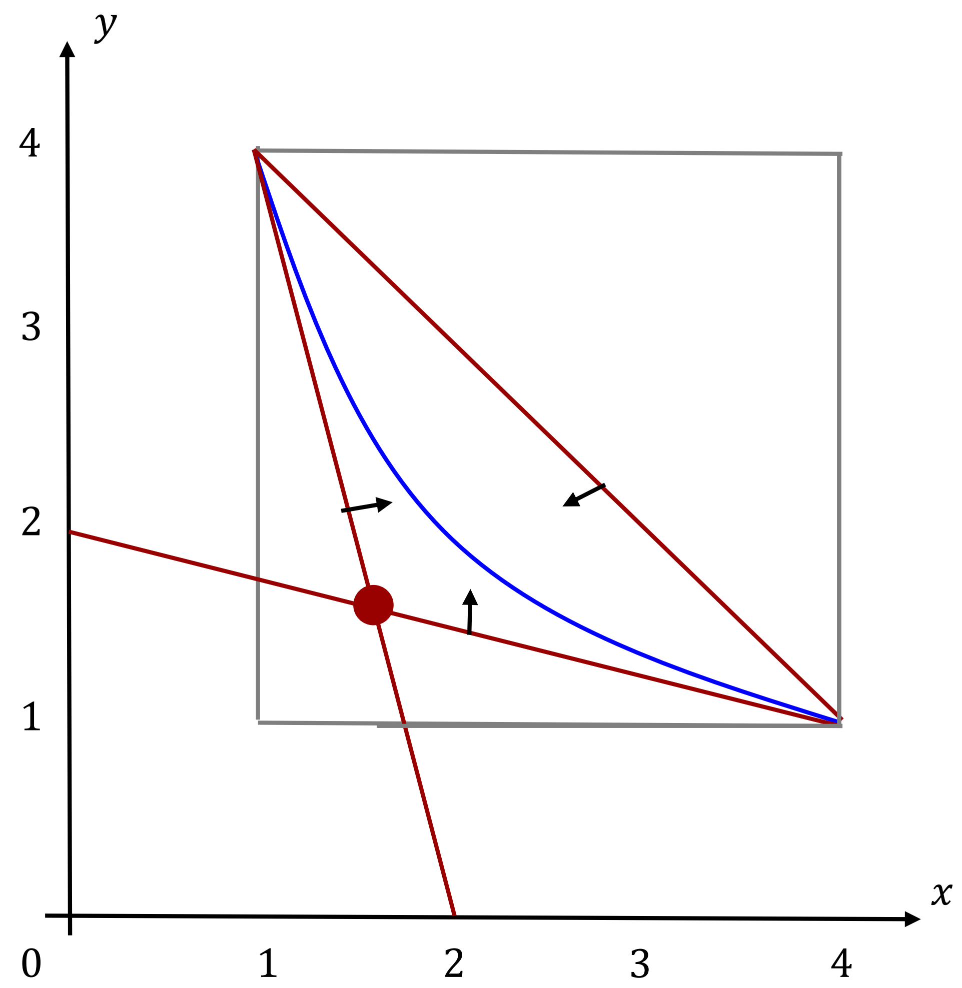
$$\text{s.t.} \quad \begin{aligned} y + x - 1 &\leq 4 \\ 4y + 4x - 16 &\leq 4 \\ 4y + x - 4 &\geq 4 \\ y + 4x - 4 &\geq 4 \\ x &\in [1,4] \\ y &\in [1,4] \end{aligned}$$

Feasible space in the lower bounding problem: area inside the triangle defined by red lines

LP solution:
$x^* = 1.6, y^* = 1.6, x^* + y^* = 3.2$

The objective function value cannot be lower than 3.2

$LB = 3.2$



B&B Example

$$\min \quad x + y$$

$$\text{s.t.} \quad xy = 4, \quad x \in [1,4], \quad y \in [1,4] \qquad x^{\mathrm{L}} = 1, x^{\mathrm{U}} = 4$$

Step 2. Upper Bounding

We evaluate the true solution at $x = 1.6$

$$y = \frac{4}{x} = 2.5, x + y = 3.1$$

B&B Example

$$\min \quad x + y$$

$$\text{s.t.} \quad xy = 4, \quad x \in [1,4], \quad y \in [1,4]$$

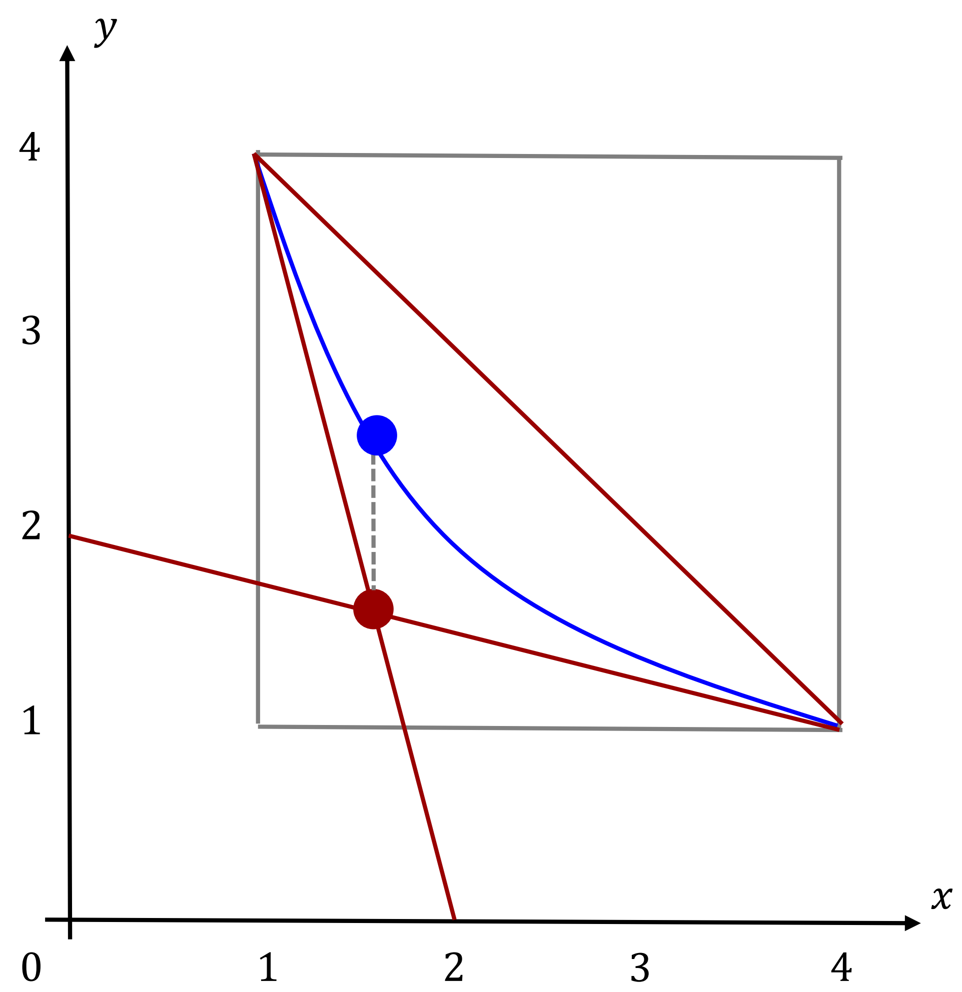
Step 2. Upper Bounding

We evaluate y at $x = 1.6$

$y = \frac{4}{x} = 2.5$, $x + y = 4.1$

The objective function value of the optimal solution cannot be (strictly) higher than 4.1

UB = 4.1



B&B Example

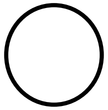
min $\quad x + y$

$$xy = 4$$

s.t. $\quad x \in [1,4] \qquad\qquad x^{\mathrm{L}} = 1, x^{\mathrm{U}} = 4$

$\quad y \in [1,4]$

Root Node



$x \in [1,4]$

$y \in [1,4]$

$LB = 3.2$

$UB = 4.1$

Step 2. Upper Bounding

We evaluate the true solution at $x = 1.6$

$$y = \frac{4}{x} = 2.5, x + y = 4.1$$

B&B Example

min $\quad x + y$

s.t. $\quad xy = 4$

$x \in [1,4]$

$y \in [1,4]$

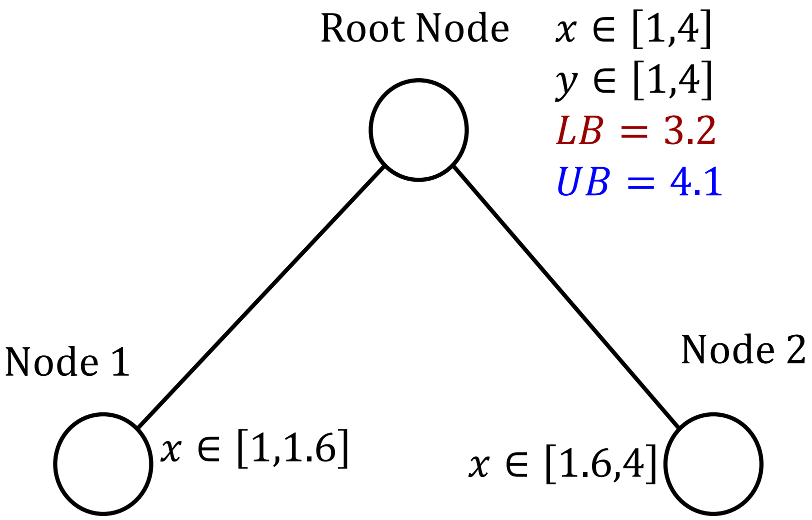
$x^{\mathrm{L}} = 1, x^{\mathrm{U}} = 4$

Step 3. Variable Subdivision

Current solution: $x^* = 1.6$

We subdivide x into 2 regions:

$x \in [1,1.6]$ and $x \in [1.6,4]$



B&B Example

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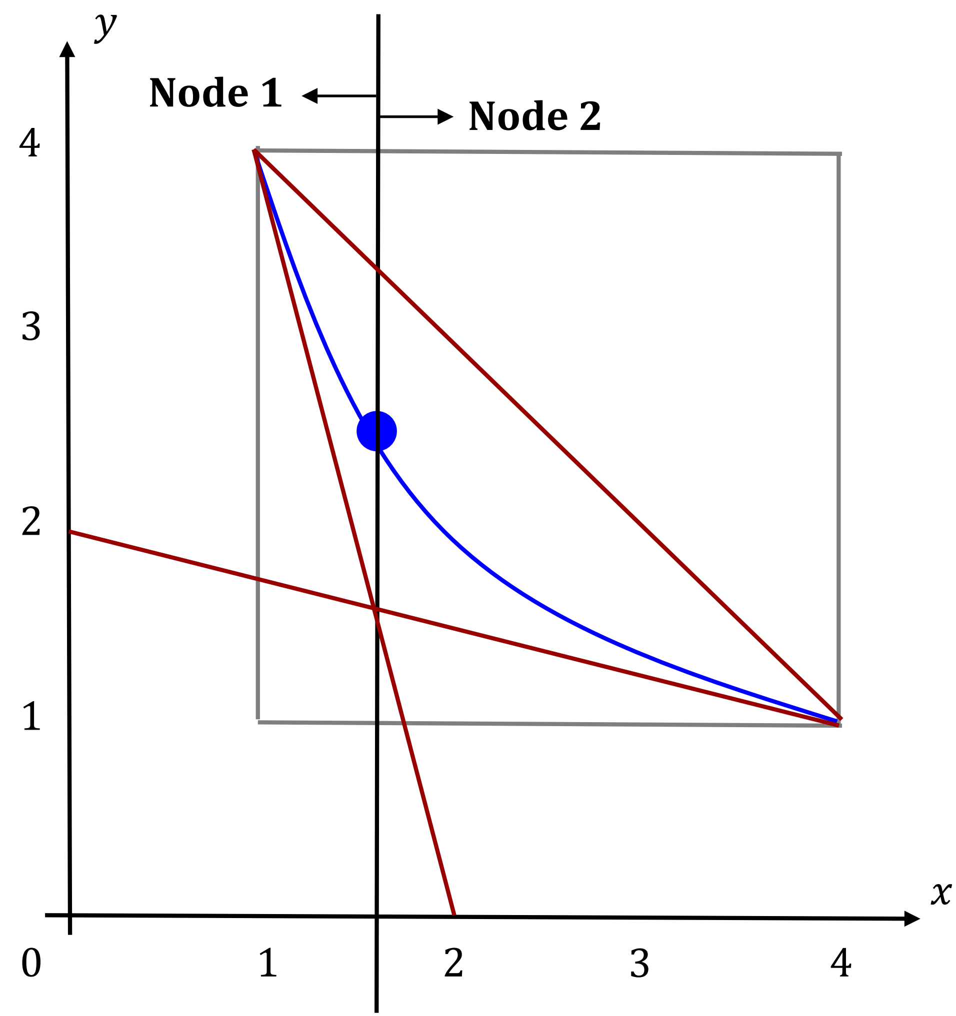


B&B Example

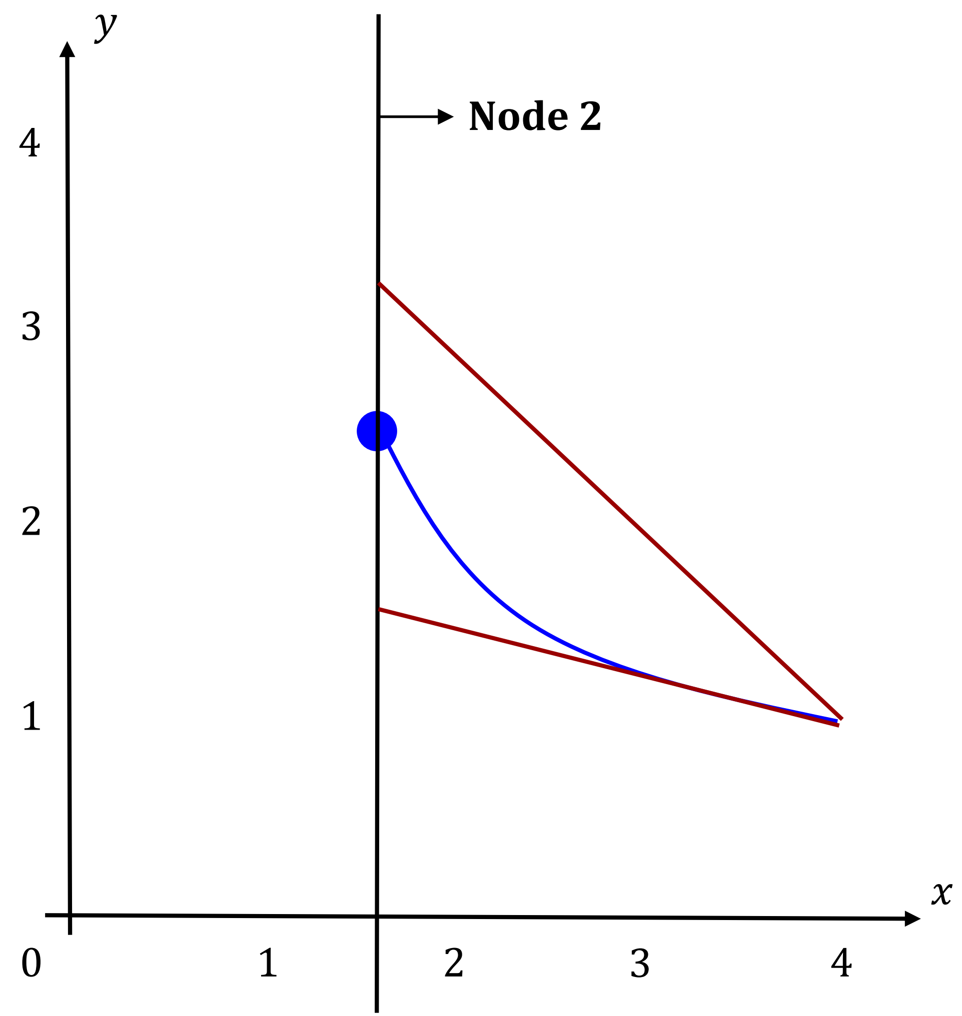
$$\min \quad x + y$$

$$\text{s.t.} \quad xy = 4, \quad x \in [1,4], \quad y \in [1,4]$$

Step 3. Variable Subdivision

Optional: Node Tightening

$x \in [1.6,4]$, $y \in ?$



B&B Example

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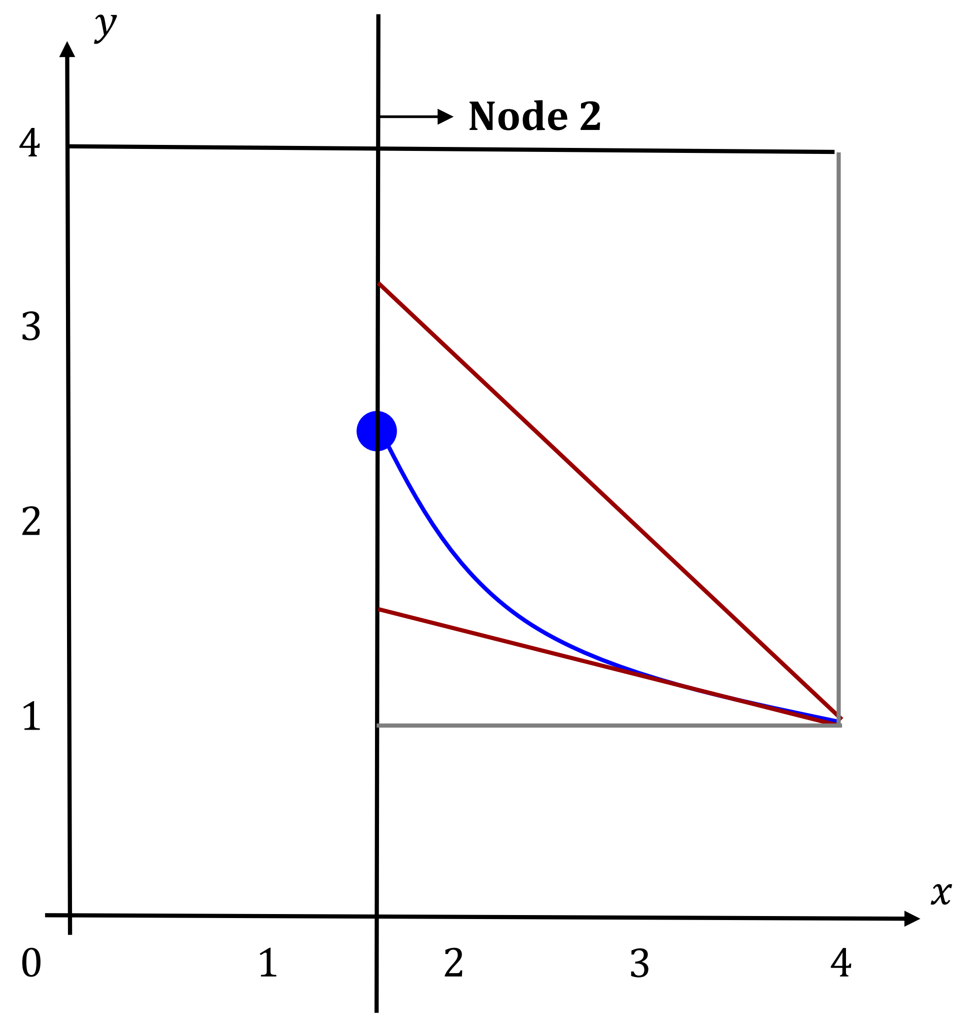
Step 3. Variable Subdivision

Optional: Node Tightening

$x \in [1.6,4]$, y ∈?

No tightening (use given bound)
y ≤ 4



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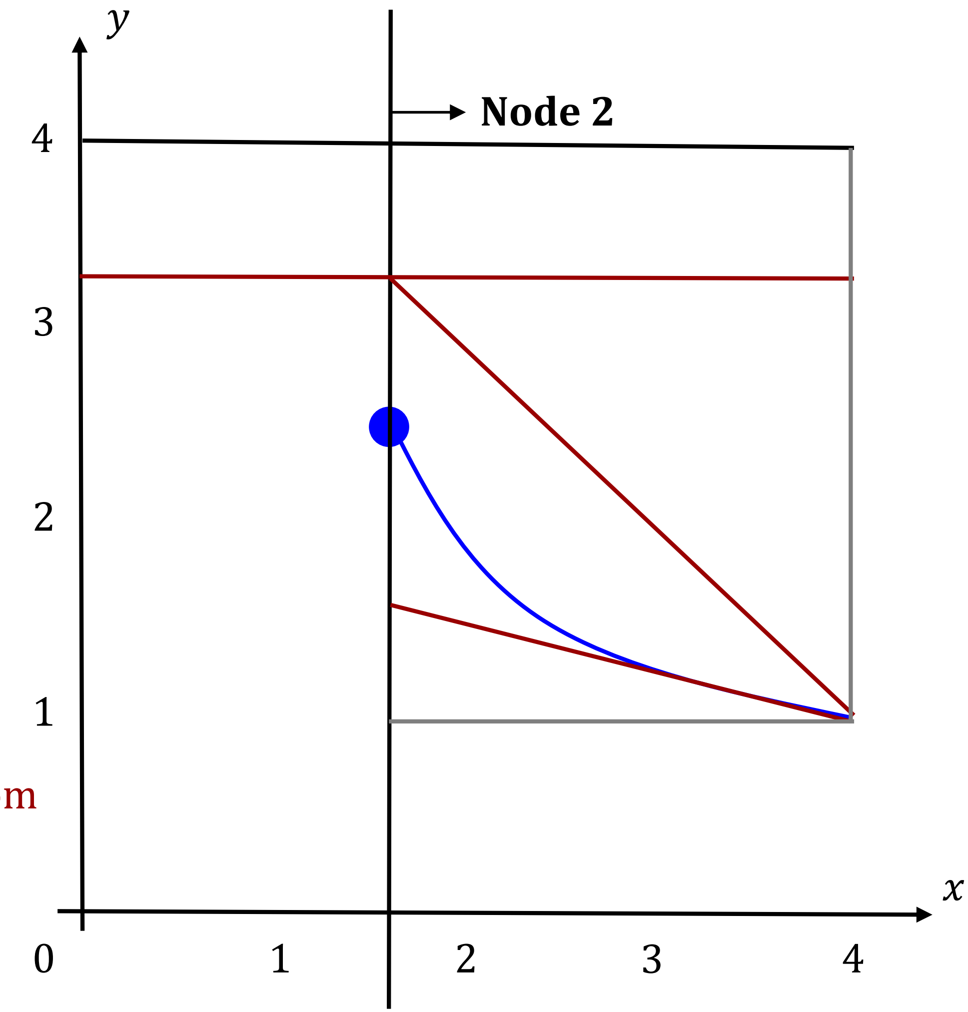
$y \leq 4$

Using (linear constraint) information from root node:

$y \leq 3.4$



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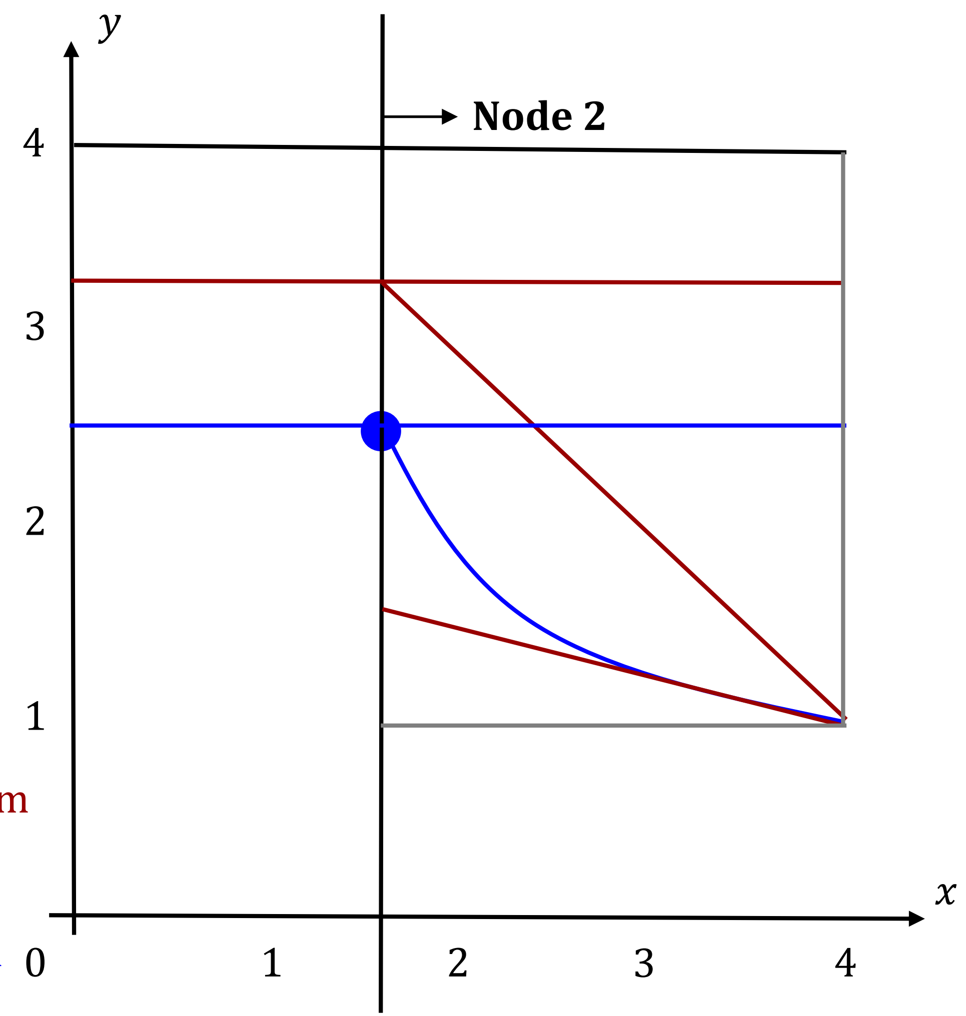
Optional: Node Tightening

$x \in [1.6,4]$, y ∈?

No tightening (use given bound)
$y \leq 4$

Using (linear constraint) information from root node:
$y \leq 3.4$

Using (nonlinear constraint) information from the problem:
$y \leq 2.5$



More on Computing

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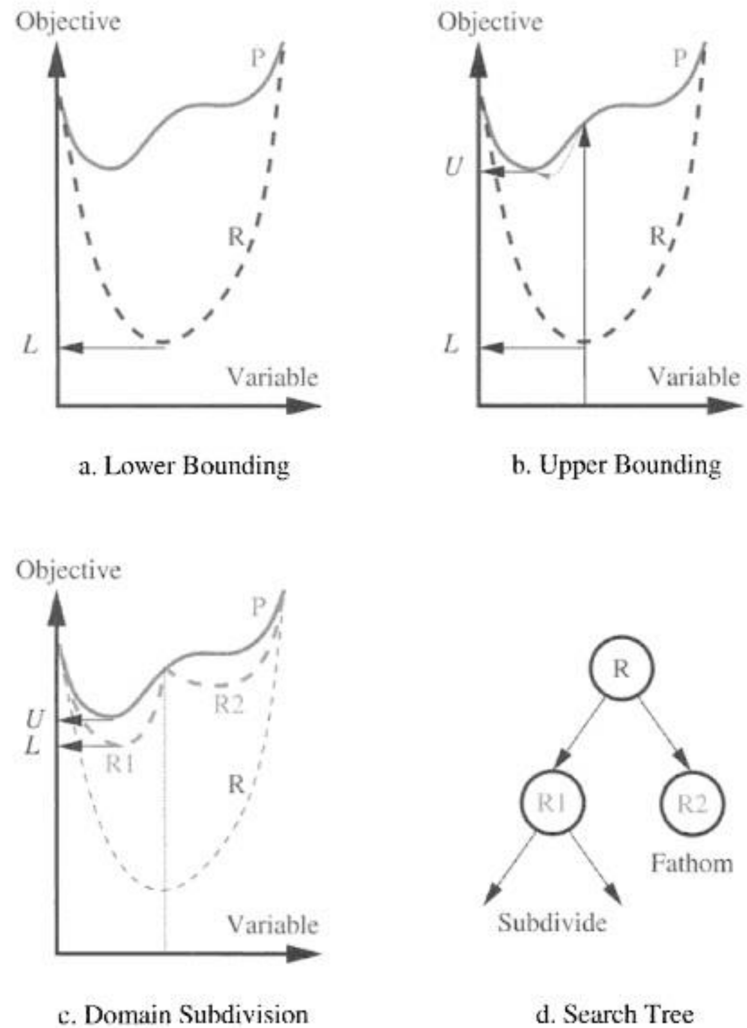

Figure 1.4: The principles of branch-and-bound

Tawarmalani and Sahinidis, 2002

Reference

Problem statement, different variances, formulations and solution methods:

Misener and Floudas, 2009
Gupte, 2012

Industrial application:

Lasdon and Waren, 1985