Global Optimization for Nonconvex Problems

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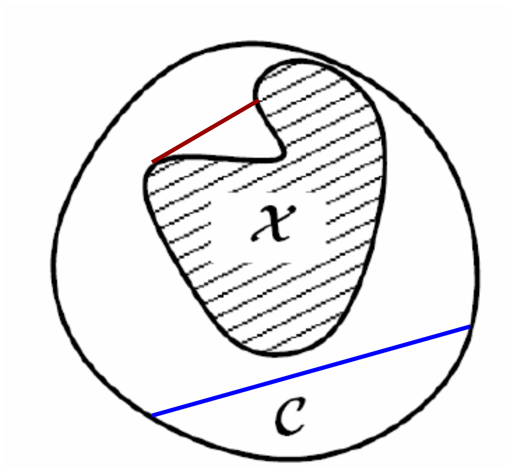
Introduction

- **Convex/Nonconvex Set**



- **Convex/Nonconvex Optimization Problem**

$min\ f(x)$ Linear obj. function

$s.t.\ x \in C$ Convex set

A local optimal solution is globally optimal

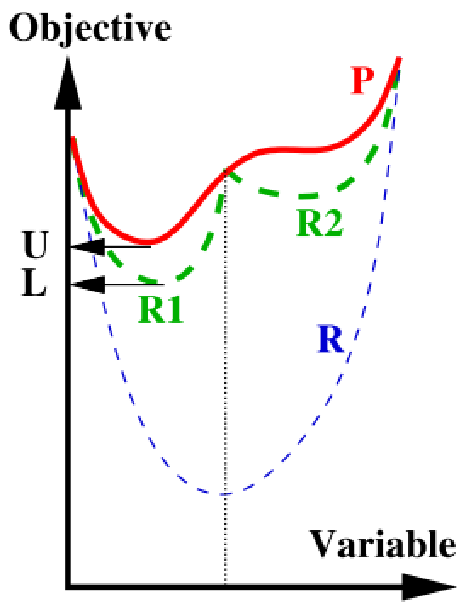
$min\ f(x)$

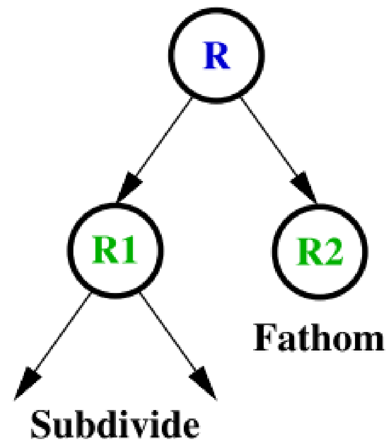
$s.t.\ x \in \chi$ Nonconvex set

Global Optimization Algorithm

Branch-and-bound:

- Implicit enumeration via a tree search
- Divide-and-conquer idea





Upper bound (U): Best found
Lower bound (L): Best possible

$$Opt.\,Gap = \left|\frac{L-U}{L}\right|$$

Example

$$\min \quad z$$

$$s.t \quad \begin{aligned} -6x_1 + 8x_2 &\le 3 \\ 3x_1 - x_2 &\le 3 \\ z &\ge -x_1 - x_2 + x_1 x_2 \\ x_1, x_2 &\in [0,5] \end{aligned}$$

Nonconvex problem

Consider 2 points:

$x_1 = 1, x_2 = 1, z = -1$ Feasible

$x_1 = 7/6, x_2 = 1/2, z = -13/12$ Feasible

Their midpoint:

$x_1 = 13/12, x_2 = 3/4, z = -25/24$ Infeasible

Step 1: Construct a convex relaxation

$$\min \quad z$$

$$\text{s.t.} \quad \begin{aligned} -6x_1 + 8x_2 &\le 3 \\ 3x_1 - x_2 &\le 3 \\ z &\ge -x_1 - x_2 + x_3 \\ x_3 &\ge x_1^{U} x_2 + x_2^{U} x_1 - x_1^{U} x_2^{U} \\ x_3 &\ge x_1^{L} x_2 + x_2^{L} x_1 - x_1^{L} x_2^{L} \\ x_1, x_2 &\in [0,5] \end{aligned}$$

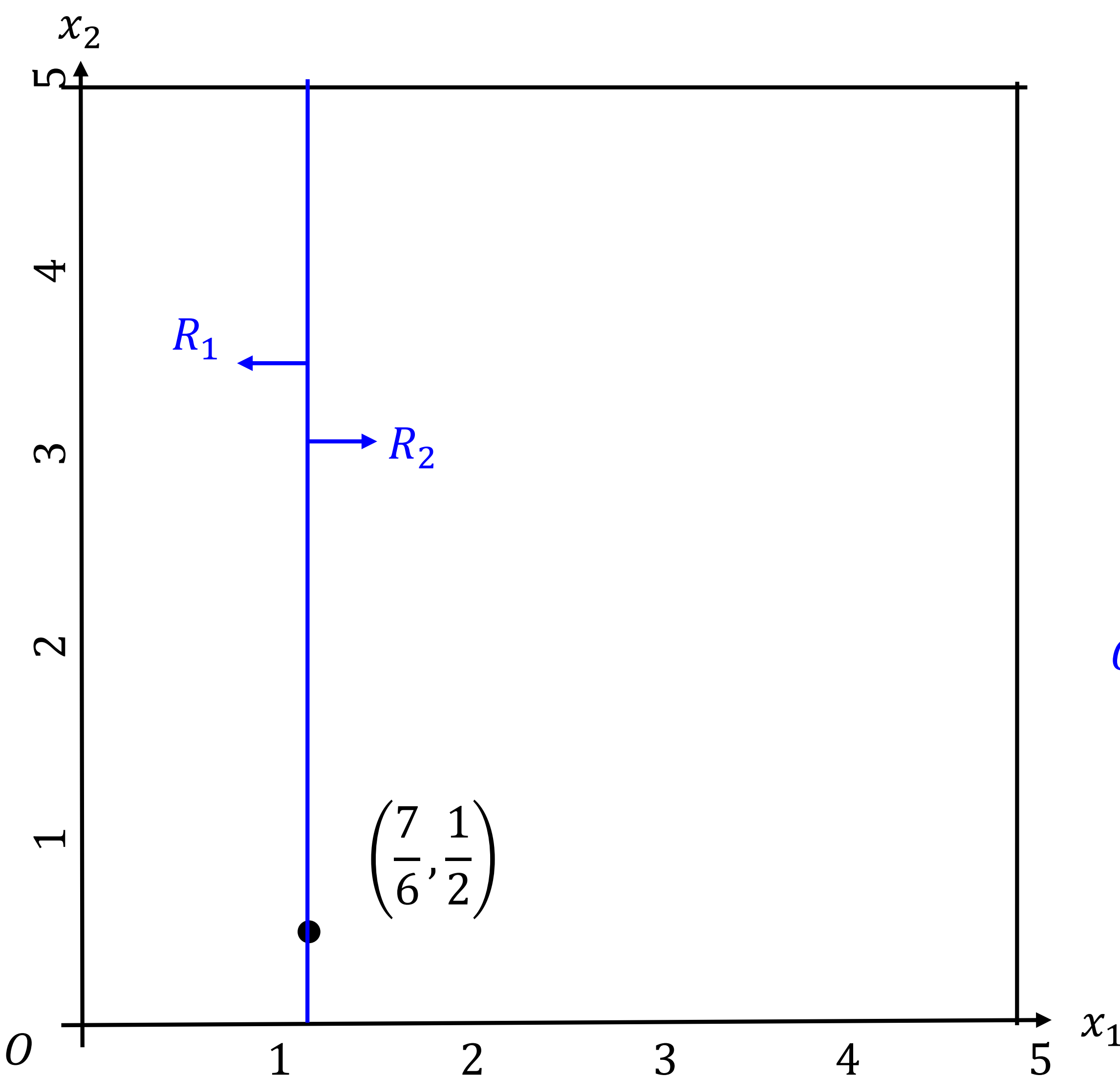
Solve the above problem, we have a valid lower bound (L) for z

$L = -3 \ (x_1 = x_2 = \frac{3}{2}, x_3 = 0)$

Solve the original problem using local methods, we have a valid upper bound (U) for z

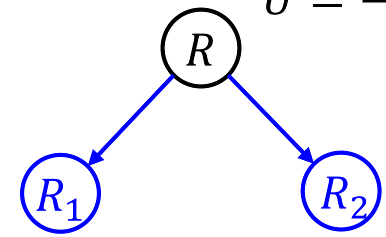
$U = -\frac{13}{12} (x_1 = \frac{7}{6}, x_2 = \frac{1}{2})$

Step 2: Branching



$L = -3$

$U = -\frac{13}{12}$

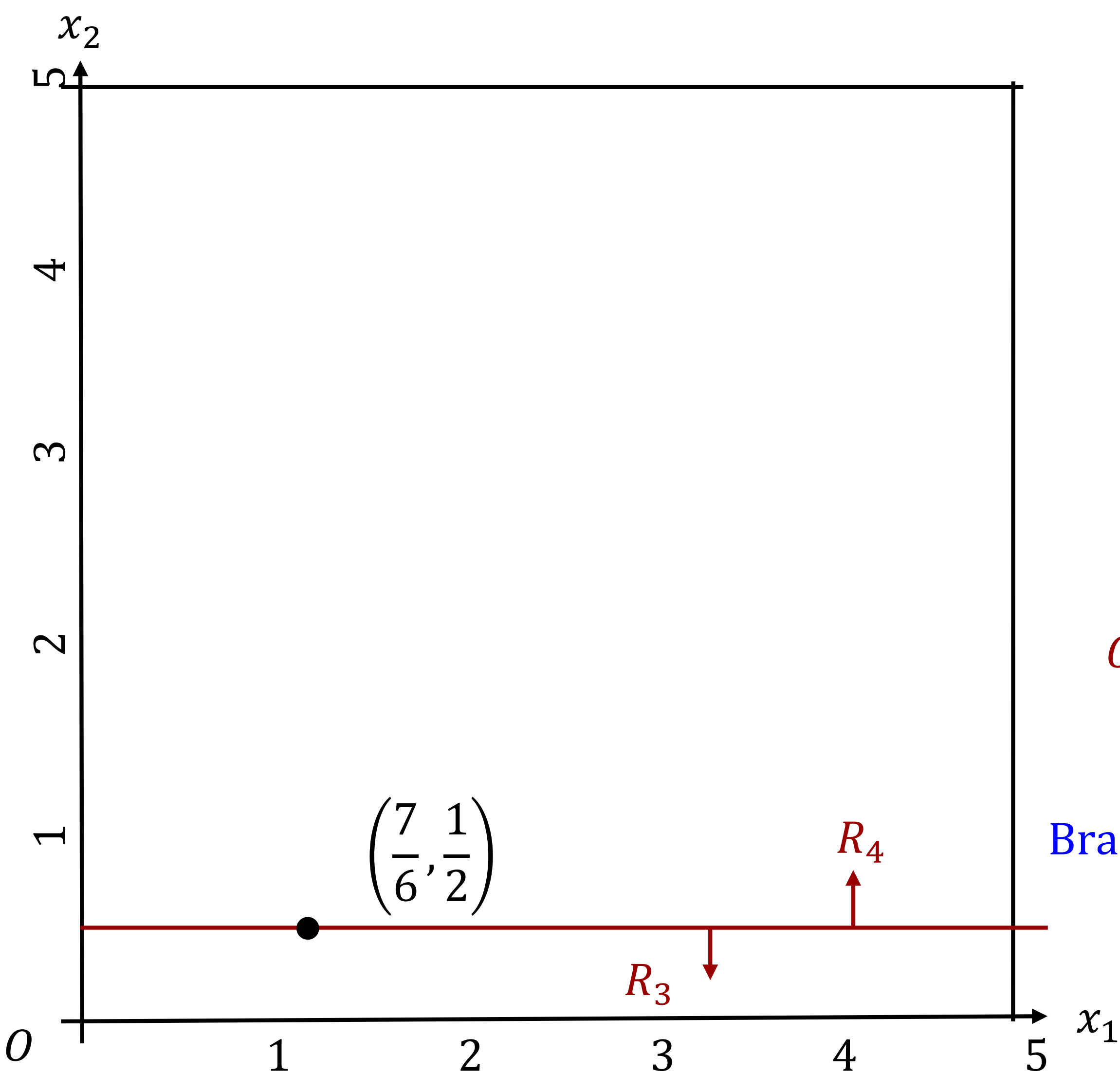


$z_{R_1} = -1.98$ $\quad z_{R_2} = -1.25$

$L = -1.98$

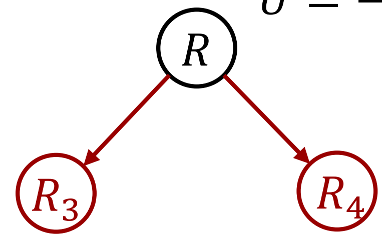
$$Opt.Gap = \frac{L - U}{L} = 45.3\%$$

Step 2: Branching



$L = -3$

$U = -\frac{13}{12}$



$z_{R_3} = -1.52$ $z_{R_4} = -2.25$

$L = -2.25$

$$Opt.Gap = \frac{L - U}{L} = 51.8\%$$

Branching decisions are important

Bound Tightening

$$\min \quad z$$

$$\text{s.t.} \quad \begin{aligned} -6x_1 + 8x_2 &\leq 3 \\ 3x_1 - x_2 &\leq 3 \\ z &\geq -x_1 - x_2 + x_3 \\ x_3 &\geq x_1^{U} x_2 + x_2^{U} x_1 - x_1^{U} x_2^{U} \\ x_3 &\geq x_1^{L} x_2 + x_2^{L} x_1 - x_1^{L} x_2^{L} \\ x_1, x_2 &\in [0,5] \end{aligned}$$

While 0 and 5 are valid bounds, can we find tighter bounds?

Bound Tightening

$$\min \quad x_1$$

$$\text{s.t.} \quad \begin{aligned} -6x_1 + 8x_2 &\leq 3 \\ 3x_1 - x_2 &\leq 3 \\ z &\geq -x_1 - x_2 + x_3 \\ x_3 &\geq x_1^{\mathrm{U}} x_2 + x_2^{\mathrm{U}} x_1 - x_1^{\mathrm{U}} x_2^{\mathrm{U}} \\ x_3 &\geq x_1^{\mathrm{L}} x_2 + x_2^{\mathrm{L}} x_1 - x_1^{\mathrm{L}} x_2^{\mathrm{L}} \\ x_1^{\mathrm{U}} &= x_2^{\mathrm{U}} = 5, x_1^{\mathrm{L}} = x_2^{\mathrm{L}} = 0 \end{aligned}$$

Optimality Based Bound Tightening (OBBT)

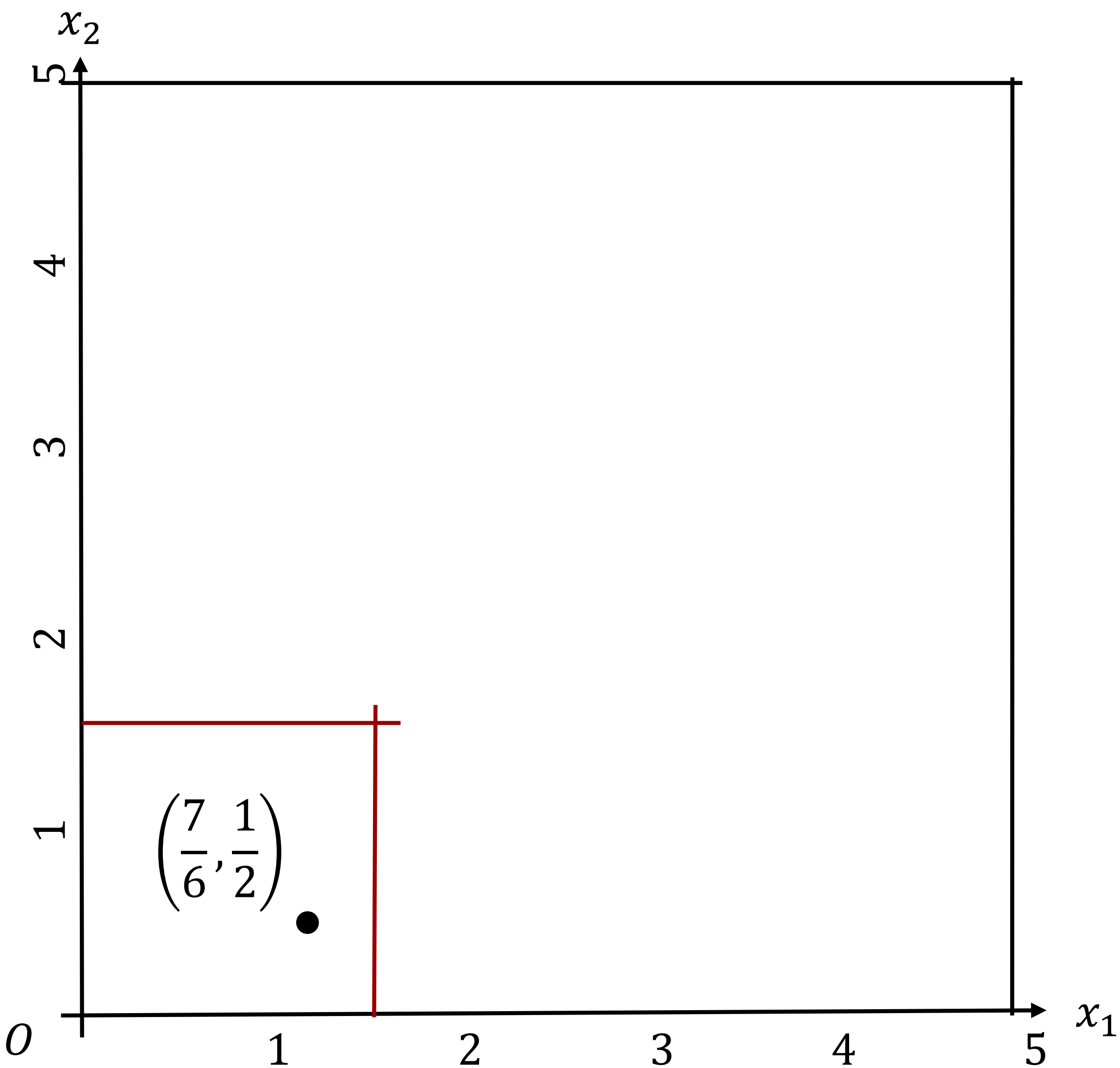
Solve 4 LPs for upper/lower bounds for x_1 and x_2, we have:
$x_1 \in [0,1.5], x_2 \in [0,1.5]$

Root Node after Bound Tightening

$$\min \quad z$$

$$s.t \quad \begin{aligned} -6x_1 + 8x_2 &\leq 3 \\ 3x_1 - x_2 &\leq 3 \\ z &\geq -x_1 - x_2 + x_3 \\ x_3 &\geq x_1^{\mathrm{U}} x_2 + x_2^{\mathrm{U}} x_1 - x_1^{\mathrm{U}} x_2^{\mathrm{U}} \\ x_3 &\geq x_1^{\mathrm{L}} x_2 + x_2^{\mathrm{L}} x_1 - x_1^{\mathrm{L}} x_2^{\mathrm{L}} \\ x_1 &\in [0,1.5], x_2 \in [0,1.5] \end{aligned}$$

Nodes after Tightening



$L = -1.5$

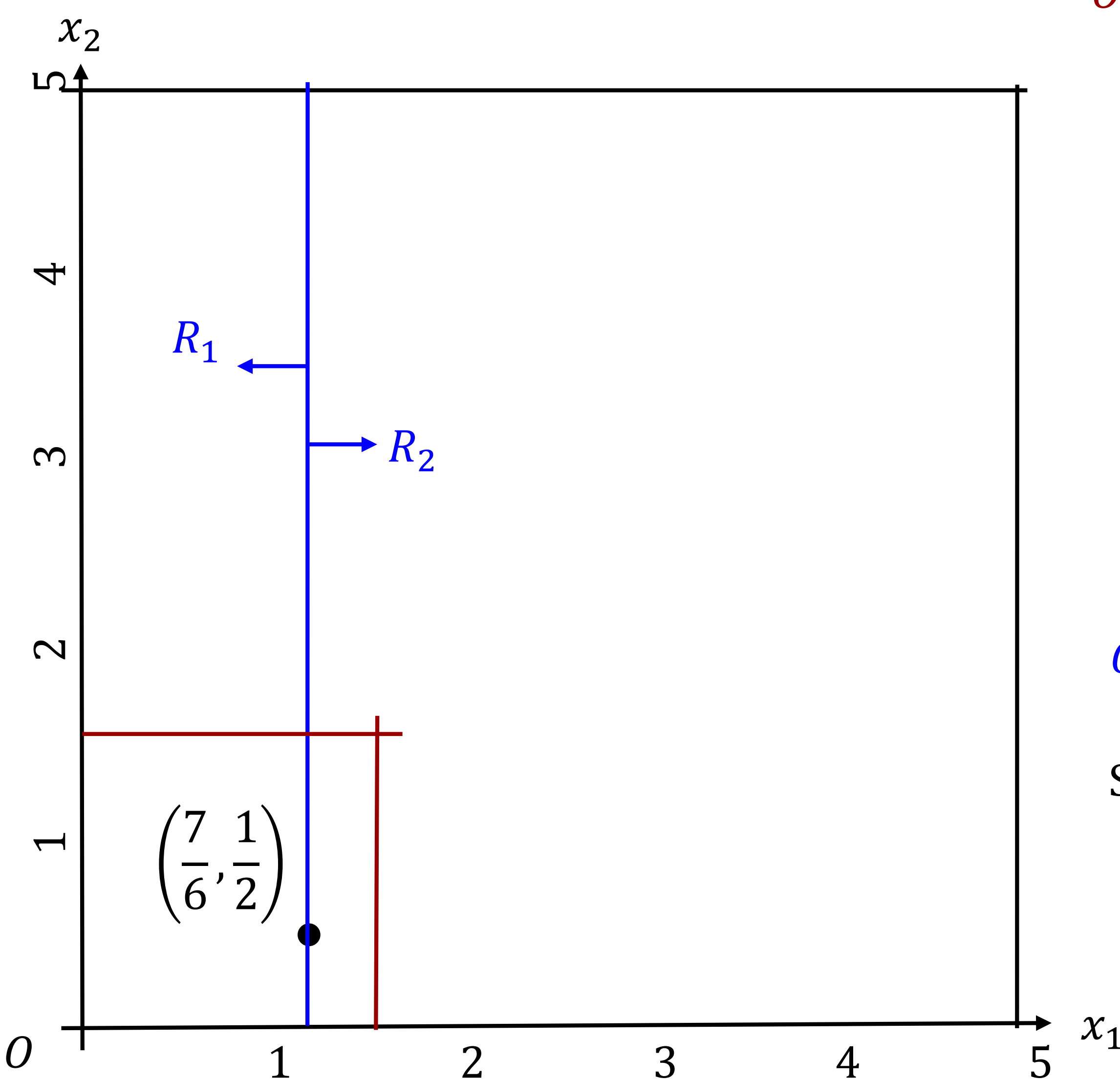
$U = -\frac{13}{12}$

R

$Opt.Gap = 38.5\%$

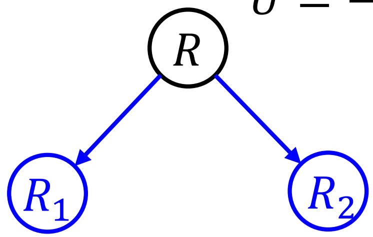
Opt. gap at the root node (with tightening) is already smaller than the opt. gap after 3 nodes without tightening (45.3% and 51.8%, resp.)

Nodes after Tightening



$Opt.Gap = 38.5\%$ $L = -1.5$

$U = -\frac{13}{12}$



$z_{R_1} = -1.34$ $z_{R_2} = -1.15$

$L = -1.34$

$$Opt.Gap = \frac{L - U}{L} = 19.2\%$$

Substantial improvement

Summary

- We introduce general idea for global optimization
- Branching decisions are important in global optimization algorithm
- Bound tightening can lead to substantial improvement for algorithm performance

Questions?